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IV. *On the Strains in the Interior of Beams.*

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Received November 6,—Read December 11, 1862.

I have long desired to possess a theory which should enable me to express and to compute numerically the actual strain or strains upon every point in the interior of a beam or girder, under circumstances analogous to those which occur in ordinary engineering applications,—partly for information on the amount of force actually sustained by the different particles of the cast or wrought iron in a solid beam, partly as a guide in the construction of lattice-bridges. The memoirs and treatises on the theories of elasticity and strains, to which I have referred, have given me no assistance*. I have therefore constructed a theory, in a form which (I believe) is new, which solves completely the problems that I had proposed to myself, and which, as I think, may, with due attention to details, be applied to all the cases that are likely to present themselves as interesting. This theory, with some of its first applications, I ask leave to place before the Royal Society.

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1. It is supposed, in the following investigations, that the beam consists of one lamina in a vertical plane,—the idea of a solid beam being supplied by the conception of a multitude of such laminæ side by side, all subject to similar strains, and therefore exerting no force one upon another. It is also supposed that the thickness of the lamina is uniform, and that its form is rectangular, the depth of the beam being equal throughout: these suppositions are made only for the sake of simplicity, as there does not appear to be any difficulty of principle in applying the theory to cases not restricted by these conditions, although the complexity would be much increased. It also appears necessary to suppose that the material of the beam yields equally, with equal forces, in different directions. Another physical supposition, which appears to be necessary for complete solution of the problem, will be stated when we reach the discussion of the first instance.

2. It is to be remarked that our theory is not intended to take account of all the strains possible in a beam, but only of those which are introduced by the weight of the beam or its load in the position in which it is used. A beam, whether of cast iron or of wrought iron, is, by the process of its manufacture, in most instances affected by permanent strains; so that, while the lamina is lying on its flat side, some parts are ready to

* Since completing this essay, I have found that considerable progress had been made in the case of figure 5, by Professor W. J. M. Rankine.

burst asunder, while others are severely compressed. When the lamina is placed in a vertical plane, these accidental strains will be combined with the strains which are produced by the weight of the beam, &c.; nevertheless our attention will be confined strictly to the latter. The algebraical expression of this idea is, that we do not want complete solutions of our differential equations; we only want solutions which will satisfy those equations; and among solutions which possess this property, we may have respect to the laws of pressure antecedently known from simpler investigations.

3. For the unit of force we shall use the weight of a unit of surface of the lamina; but in writing the expressions, we shall omit the word "weight," as no ambiguity can be produced by its absence. For the unit of the force of compression, or of tension (which is merely compression with changed sign, or negative compression), we must refer to such considerations as the following. A force of tension is not a force acting in a single line; it is a force acting in parallel or nearly parallel lines, with nearly constant magnitude over a considerable extent of surface. In a large structure, like the Britannia Bridge for instance, on any space one inch broad there is a certain force of tension; but on the neighbouring space of one inch broad there is the same force of tension, and so for each inch in a long succession there is sensibly the same force of tension. The force of tension, acting on a certain breadth measured perpendicularly to the direction of tension, will therefore be proportional to that breadth, or will be equal to the weight of a surface or ribbon whose breadth is the breadth which sustains the action, and whose length varies with the magnitude of the tension. That length is the proper measure of tension. When the breadth subject to the action $=1$ (the unit of linear measure), the amount of action is expressed simply by that length; when the breadth has another value, the amount of action is the product of the value of breadth by the length which measures the tension. The same remarks apply to the measure of compression.

4. We must now consider the effect of tension estimated in a direction inclined at an angle φ to the direction of tension. Suppose that a cut is made through the lamina, at right angles to the direction of tension, and that the effect of tension is to separate the sides of the cut. And suppose the direction of tension to rotate in the plane of the lamina. As the rotation proceeds, the tendency to open the cut diminishes, till, when $\varphi=90°$, the tendency vanishes entirely. But when φ becomes greater than $90°$, the tendency to open the cut is restored, and when $\varphi=180°$, it is exactly as great as when $\varphi=0$. As φ is further increased, the tendency diminishes by the same degrees, and vanishes for $\varphi=270°$; then increases till $\varphi=360°$. It is never converted into a force of compression, and its changes are the same for positive and for negative changes of φ. These considerations show that the effect must be represented by a formula containing only even powers of $\cos\varphi$. And the following consideration will show that there will be only one term, multiplying $\cos^2\varphi$. When the tension acts at right angles to the cut, if t be the length which measures the tension, and if l be the length of a portion of the cut, the force which acts is the weight of the ribbon whose length is t and breadth l; and is

therefore $=lt$. But when the direction has rotated through ϕ, the force acting obliquely on l is the weight of the ribbon whose length is t and breadth $l.\cos\phi$, and is therefore $=lt.\cos\phi$. And this force is not normal to the cut, but makes the angle ϕ with the normal; and therefore the force which is normal to the cut, acting on the length l, is $lt.\cos\phi\times\cos\phi=lt.\cos^2\phi=l\times t.\cos^2\phi$. Consequently the measure of the tension, at the angle ϕ to the original tension, is $t.\cos^2\phi$. The same theorem applies to compression.

5. We must now proceed to consider the coexistence of two or more forces of compression or extension. There is no difficulty in conceiving that a plate of metal may at the same time be extended in one direction and compressed in another direction transversal to the former. But on consideration it will be found equally easy to conceive that a plate of metal may sustain at the same time several forces of compression, or of extension, or of both. It is easy to devise an apparatus which will produce these effects. Such forces may exist in the strains of a beam; and it is important to show that they can be included in a simple investigation. The following theorem is now to be proved. "Whatever be the number and directions of the forces of compression and extension, their combination may in all cases be represented by the combination of two forces at right angles,—these forces being sometimes both of compression, sometimes both of extension, sometimes one a force of extension and the other a force of compression, and generally unequal in magnitude." The following is the demonstration. Suppose that there are forces of compression (forces of extension being represented as negative forces of compression) of magnitudes A_1, A_2, &c., acting in directions which make angles α_1, α_2, &c. with a fixed line. Let us estimate the effect of their combination in a direction making any angle ψ with the same line. The angles between the directions of the several forces and this direction are respectively $\alpha_1-\psi$, $\alpha_2-\psi$, &c.; and therefore, by the last article, their effects in the direction ψ are $A_1.\cos^2(\alpha_1-\psi)$, $A_2.\cos^2(\alpha_2-\psi)$, &c.; or

$$A_1.\cos^2\alpha_1.\cos^2\psi+2A_1.\cos\alpha_1.\sin\alpha_1.\cos\psi.\sin\psi+A_1.\sin^2\alpha_1.\sin^2\psi,$$
$$A_2.\cos^2\alpha_2.\cos^2\psi+2A_2.\cos\alpha_2.\sin\alpha_2.\cos\psi.\sin\psi+A_2.\sin^2\alpha_2.\sin^2\psi,$$
$$\&c.;$$

the sum of which may be represented by

$$\Sigma(A.\cos^2\alpha).\cos^2\psi+\Sigma(2A.\cos\alpha.\sin\alpha).\cos\psi.\sin\psi+\Sigma(A.\sin^2\alpha).\sin^2\psi,$$

or

$$a.\cos^2\psi+b.\cos\psi.\sin\psi+c.\sin^2\psi;$$

where a, b, c may have any magnitude and either sign. And it is to be shown that we can find a force B acting at the angle β, and a force C acting at the angle $\beta+90°$, whose combination will produce the same effect.

Now the effect of these forces, by the theorem of last article, is

$$B.\cos^2(\beta-\psi)+C.\cos^2(\beta+90°-\psi),$$

or

$$B.\cos^2\beta.\cos^2\psi+2B.\cos\beta.\sin\beta.\cos\psi.\sin\psi+B.\sin^2\beta.\sin^2\psi$$
$$+C.\sin^2\beta.\cos^2\psi-2C.\sin\beta.\cos\beta.\cos\psi.\sin\psi+C.\cos^2\beta.\sin^2\psi.$$

Comparing this, term by term, with the former,

$$\mathrm{B}\,.\cos^2\beta+\mathrm{C}\,.\sin^2\beta=a;$$
$$\mathrm{B}\,.\sin^2\beta+\mathrm{C}\,.\cos^2\beta=c;$$
$$(\mathrm{B}-\mathrm{C})\,.\sin 2\beta \qquad =b.$$

The difference of the first and second equations gives

$$(\mathrm{B}-\mathrm{C})\,.\cos 2\beta=a-c;$$

and the quotient of the third by this gives

$$\tan 2\beta=\frac{b}{a-c};$$

which always gives a possible value for β.

Then $\mathrm{B}-\mathrm{C}=\frac{b}{\sin 2\beta}$ or $=\frac{a-c}{\cos 2\beta}$, which is always possible. And, by adding the first and second equations,

$$\mathrm{B}+\mathrm{C}=a+c.$$

By the combination of B+C and B−C, B and C are found. Thus all the elements may be found, for representing the effect of any number of forces of compression or extension, by the effect of two forces of compression or extension acting at right angles to each other. Our succeeding investigations therefore will be confined to the consideration of two such forces acting at each point.

We are now in a state to proceed with the consideration of the strains in a beam.

6. In fig. 1, Plate V., let the parallelogram represent a beam, supported in any way, as for instance by having one end fixed into a wall, and subject to any force, as for instance the vertical reaction R of a support at distance h. If R is negative, it will represent a weight hanging on the beam. Conceive a line to pass in any curved or crooked direction, from the lower to the upper edge, dividing the beam into two parts, a near part and a distant part. This division is to be understood merely as a line visible to the eye; it is not to be contemplated as a mechanical separation; for if it were such, the metal on one side could be considered as acting upon the metal on the other side only in the direction perpendicular to the separating line; which action, in many cases (as when the separating line is vertical), would obviously be incompetent to support the distant part of the beam. The compressions and tensions, which we can suppose to exist while the continuity is mechanically uninterrupted, will suffice (with or without other forces) to support the distant part. Now if the upper end of the curve terminates in the upper edge of the beam, conceive the curve to continue along that edge till it meets the upper angle at the end of the beam; if it terminates in the vertical end of the beam, conceive it carried upwards till it meets the upper angle; thus the special actions which sometimes operate in the limiting lines will be separated from those in the dividing curve. Let r and s be the length and depth of the beam; x the horizontal abscissa (measured from o), and y the vertical ordinate (measured from the lower edge)

of any point of the curve. At the first limit of the curve, the coordinates are z, 0; at the last, the coordinates are r, s.

7. The distant part of the beam is supported by the forces of compression (this term, with negative values, including tensions) across every part of the curve, combined with the reaction R. At the point whose coordinates are x, y, conceive that there is one force of compression B whose direction makes the angle β to the left side of y produced, and another force of compression C whose direction makes the angle $\beta+90°$ to the left side of y produced. And, in figure 2, consider the actions of these on the small element δs of the curve, or rather the actions on a portion of the lamina, including δs. Let θ be the angle made by δs with y. The direction of the action of B makes with δs the angle $\beta+\theta$; and therefore the breadth of the ribbon representing its action is $\delta s\times\sin(\beta+\theta)$, and its whole force is $\mathrm{B}\,.\,\delta s\times\sin(\beta+\theta)$. Resolving this in the directions of x and y, we have for the effects of B on the distant part of the beam,

$$\text{In the direction } x,\quad \mathrm{B}\,.\,\delta s\times\sin(\beta+\theta)\times\sin\beta,$$
$$\text{In the direction } y,\quad -\mathrm{B}\,.\,\delta s\times\sin(\beta+\theta)\times\cos\beta.$$

In like manner, the effects of C on the distant part of the beam are,

$$\text{In the direction } x,\quad \mathrm{C}\,.\,\delta s\times\sin(\beta+90°+\theta)\times\sin(\beta+90°),$$
$$\text{In the direction } y,\quad -\mathrm{C}\,.\,\delta s\times\sin(\beta+90°+\theta)\times\cos(\beta+90°).$$

Expanding the sine, we have, for the whole force in the direction x,

$$\{\mathrm{B}\,.\,\sin^2\beta+\mathrm{C}\,.\,\cos^2\beta\}\,.\,\cos\theta\,.\,\delta s+\{\mathrm{B}\,.\,\cos\beta\,.\,\sin\beta-\mathrm{C}\,.\,\sin\beta\,.\,\cos\beta\}\,.\,\sin\theta\,.\,\delta s,$$

and for the whole force in the direction y,

$$\{-\mathrm{B}\,.\,\sin\beta\,.\,\cos\beta+\mathrm{C}\,.\,\cos\beta\,.\,\sin\beta\}\,.\,\cos\theta\,.\,\delta s+\{-\mathrm{B}\,.\,\cos^2\beta-\mathrm{C}\,.\,\sin^2\beta\}\,.\,\sin\theta\,.\,\delta s.$$

But $\cos\theta\,.\,\delta s=\delta y$, $\sin\theta\,.\,\delta s=\delta x$. And using for convenience the following letters,

$$\mathrm{L}=\mathrm{B}\,.\,\sin^2\beta+\mathrm{C}\,.\,\cos^2\beta,$$
$$\mathrm{M}=(\mathrm{B}-\mathrm{C})\,.\,\sin\beta\,.\,\cos\beta,$$
$$\mathrm{Q}=-\mathrm{B}\,.\,\cos^2\beta-\mathrm{C}\,.\,\sin^2\beta,$$

we have for the whole forces on the element δs,

$$\text{In the direction } x,\quad \mathrm{L}\,.\,\delta y+\mathrm{M}\,.\,\delta x,$$
$$\text{In the direction } y,\quad -\mathrm{M}\,.\,\delta y+\mathrm{Q}\,.\,\delta x.$$

It must be borne in mind that the force in direction x acts in a line whose vertical ordinate is y, and that the force in direction y acts in a line whose horizontal ordinate is x.

8. There is another force acting on this portion of the distant part, namely, the weight of the lamina included between the ordinates corresponding to x and $x+\delta x$; which, estimated in the direction y, is $-y\,.\,\delta x$, acting in a line whose horizontal ordinate is x.

And, besides these forces which act at every point of the curve, there is the reaction $+\mathrm{R}$ in the direction y, acting in a line whose horizontal ordinate is h.

9. We have now collected all the elements for the equations of equilibrium of the distant part of the beam, and we proceed to form those equations. For δy we shall put $p.\delta x$. The equations are as follows:

First, equation for forces in x:

$$\int dx.(\mathrm{L}p+\mathrm{M})=0. \qquad (1.)$$

Second, equation for forces in y:

$$\int dx.(-\mathrm{M}p+\mathrm{Q}-y)+\mathrm{R}=0. \qquad (2.)$$

Third, equation of moments:

$$\int dx.(\mathrm{L}yp+\mathrm{M}y+\mathrm{M}xp-\mathrm{Q}x+xy)-\mathrm{R}h=0. \qquad (3.)$$

It will be convenient at once to make $y-\mathrm{Q}=\mathrm{O}$; and the equations become

$$\int dx.(\mathrm{L}p+\mathrm{M})=0. \qquad (4.)$$

$$\int dx.(\mathrm{M}p+\mathrm{O})-\mathrm{R}=0. \qquad (5.)$$

$$\int dx.(\mathrm{L}yp+\mathrm{M}y+\mathrm{M}xp+\mathrm{O}x)-\mathrm{R}h=0. \qquad (6.)$$

10. We shall now introduce a consideration which will prove singularly advantageous for the solution of these equations. Referring to figure 3, the equations which we have obtained apply to the curve $a\,b\,c\,d\,e\,f$. The same equations, *mutatis mutandis*, apply to the curve $a\,b\,g\,d\,e\,f$. Hence the variations in those equations produced by passing from one of these curves to the other will $=0$. Now these variations are clearly such as are treated in the Calculus of Variations. We may therefore form the variations of the equations according to the rules of the Calculus of Variations, and equate those variations to zero. R and Rh will disappear.

11. The left side of equations (4.), (5.), (6.), is in each case a function of x, y, p (L, M, and O depending on the position of the point in the lamina, and therefore being functions of x and y), and of no other differential coefficients. Therefore the equation of variations in each case, in the usual language of the Calculus of Variations, will have the form $\mathrm{N}-\frac{d(\mathrm{P})}{dx}=0$. Applying this in each instance we have;—

For $\delta.\int dx.(\mathrm{L}p+\mathrm{M})$:

$$\mathrm{N}=\frac{d\mathrm{L}}{dy}p+\frac{d\mathrm{M}}{dy}; \qquad \mathrm{P}=\mathrm{L}; \qquad \frac{d(\mathrm{P})}{dx}=\frac{d\mathrm{L}}{dx}+\frac{d\mathrm{L}}{dy}p;$$

therefore

$$\frac{d\mathrm{L}}{dy}p+\frac{d\mathrm{M}}{dy}-\frac{d\mathrm{L}}{dx}-\frac{d\mathrm{L}}{dy}p=0,$$

or

$$\frac{d\mathrm{M}}{dy}-\frac{d\mathrm{L}}{dx}=0. \qquad (7.)$$

For $\delta.\int dx(\mathrm{M}p+\mathrm{O})$: in the same manner,

$$\frac{d\mathrm{O}}{dy}-\frac{d\mathrm{M}}{dx}=0. \qquad (8.)$$

For $\delta.\int dx(\mathrm{L}yp+\mathrm{M}y+\mathrm{M}xp+\mathrm{O}x)$:

$$\mathrm{N}=\frac{d\mathrm{L}}{dy}yp+\mathrm{L}p+\frac{d\mathrm{M}}{dy}y+\mathrm{M}+\frac{}{dy}xp+\frac{d\mathrm{O}}{dy}x;\qquad \mathrm{P}=\mathrm{L}y+\mathrm{M}x;$$

$$\frac{d(\mathrm{P})}{dx}=\frac{d\mathrm{L}}{dx}y+\frac{d\mathrm{L}}{dy}py+\mathrm{L}p+\frac{d\mathrm{M}}{dx}x+\frac{d\mathrm{M}}{dy}px+\mathrm{M};$$

therefore

$$\frac{d\mathrm{L}}{dy}yp+\mathrm{L}p+\frac{d\mathrm{M}}{dy}y+\mathrm{M}+\frac{d\mathrm{M}}{dy}xp+\frac{d\mathrm{O}}{dy}x-\frac{d\mathrm{L}}{dx}y-\frac{d\mathrm{L}}{dy}py-\mathrm{L}p-\frac{d\mathrm{M}}{dx}x-\frac{d\mathrm{M}}{dy}px-\mathrm{M}=0,$$

or

$$y\left(\frac{d\mathrm{M}}{dy}-\frac{d\mathrm{L}}{dx}\right)+x\left(\frac{d\mathrm{O}}{dy}-\frac{d\mathrm{M}}{dx}\right)=0.$$

This equation, by virtue of equations (7.) and (8.), is identically true, and therefore adds nothing to our knowledge. The information, then, that we have obtained from our process is comprised in the two equations

$$\frac{d\mathrm{M}}{dy}=\frac{d\mathrm{L}}{dx}; \quad \ldots\ldots\ldots \quad (7.)$$

$$\frac{d\mathrm{O}}{dy}=\frac{d\mathrm{M}}{dx}. \quad \ldots\ldots\ldots \quad (8.)$$

From this it follows that L, M, O are the three partial differential equations of the second order of a function F of x and y, such that

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2},\quad \mathrm{M}=\frac{d^2\mathrm{F}}{dxdy},\quad \mathrm{O}=\frac{d^2\mathrm{F}}{dx^2};$$

and we may substitute these symbols for L, M, O, in the equations of equilibrium of the distant part of the beam.

12. If it had been necessary to use expressions of the utmost possible generality, we must have said

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2}+\varphi(y),\quad \mathrm{M}=\frac{d^2\mathrm{F}}{dxdy},\quad \mathrm{O}=\frac{d^2\mathrm{F}}{dx^2}+\psi(x),$$

where the forms of the functions φ and ψ are arbitrary. Suppose now that F is so determined that the substitution of $\frac{d^2\mathrm{F}}{dy^2}$, $\frac{d^2\mathrm{F}}{dxdy}$, and $\frac{d^2\mathrm{F}}{dx^2}$ for L, M, O will satisfy the equations (4.), (5.), (6.) in their entirety. Then the substitution of $\varphi(y)$ and $\psi(x)$ alone must satisfy those equations deprived of their constant terms; and therefore $\varphi(y)$ and $\psi(x)$ may be multiplied to any degree, or different functions of the same character may be added to them. These remarks clearly indicate that these functions represent accidental strains such as we have spoken of in article 2, and they are therefore to be neglected. We confine ourselves therefore to the terms

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2},\quad \mathrm{M}=\frac{d^2\mathrm{F}}{dxdy},\quad \mathrm{O}=\frac{d^2\mathrm{F}}{dx^2}.$$

13. Making these substitutions, and restoring for pdx its original expression dy, the equations become the following:

$$\int\left(dy.\frac{d^2\mathrm{F}}{dy^2}+dx.\frac{d^2\mathrm{F}}{dxdy}\right)=0; \quad \ldots \quad (9.)$$

$$\int\left(dy.\frac{d^2\mathrm{F}}{dxdy}+dx.\frac{d^2\mathrm{F}}{dx^2}\right)-\mathrm{R}=0; \quad \ldots \quad (10.)$$

$$\int\left(dy.\frac{d^2\mathrm{F}}{dy^2}.y+dx.\frac{d^2\mathrm{F}}{dxdy}y+dy.\frac{d^2\mathrm{F}}{dxdy}x+dx.\frac{d^2\mathrm{F}}{dx^2}x\right)-\mathrm{R}h=0; \quad \ldots \quad (11.)$$

the integrals being taken from $x, y=z, 0$, to $x, y=r, s$. Now

$$\frac{d^2\mathrm{F}}{dy^2}dy+\frac{d^2\mathrm{F}}{dxdy}dx=d\left(\frac{d\mathrm{F}}{dy}\right); \qquad \frac{d^2\mathrm{F}}{dxdy}dy+\frac{d^2\mathrm{F}}{dx^2}dx=d\left(\frac{d\mathrm{F}}{dx}\right);$$

and the same symbols appear in the bracket of equation (11.). Hence the equations become

$$\int.d\left(\frac{d\mathrm{F}}{dy}\right)=0; \quad \ldots \quad (12.)$$

$$\int.d\left(\frac{d\mathrm{F}}{dx}\right)-\mathrm{R}=0; \quad \ldots \quad (13.)$$

and

$$\int\left\{y.d\left(\frac{d\mathrm{F}}{dy}\right)+x.d\left(\frac{d\mathrm{F}}{dx}\right)\right\}-\mathrm{R}h=0.$$

Integrating the quantities under the bracket by parts, the bracket becomes

$$y\frac{d\mathrm{F}}{dy}+x\frac{d\mathrm{F}}{dx}-\int\left(dy.\frac{d\mathrm{F}}{dy}+dx.\frac{d\mathrm{F}}{dx}\right).$$

But $\frac{d\mathrm{F}}{dy}dy+\frac{d\mathrm{F}}{dx}dx=d(\mathrm{F})$. The value of the bracket, therefore, is $y\frac{d\mathrm{F}}{dy}+x\frac{d\mathrm{F}}{dx}-\mathrm{F}$; and equation (11.) becomes

$$\int.d\left\{y\frac{d\mathrm{F}}{dy}+x\frac{d\mathrm{F}}{dx}-\mathrm{F}\right\}-\mathrm{R}h=0. \quad \ldots \quad (14.)$$

Attaching the subscripts z, 0 and r, s to the symbols or brackets, to denote the values which the expressions assume when z, 0 or r, s are substituted for x, y, the equations finally become

$$\left(\frac{d\mathrm{F}}{dy}\right)_{r,s}-\left(\frac{d\mathrm{F}}{dy}\right)_{z,0}=0. \quad \ldots \quad (15.)$$

$$\left(\frac{d\mathrm{F}}{dx}\right)_{r,s}-\left(\frac{d\mathrm{F}}{dx}\right)_{z,0}-\mathrm{R}=0. \quad \ldots \quad (16.)$$

$$\left\{y\frac{d\mathrm{F}}{dy}+x\frac{d\mathrm{F}}{dx}-\mathrm{F}\right\}_{r,s}-\left\{y\frac{d\mathrm{F}}{dy}+x\frac{d\mathrm{F}}{dx}-\mathrm{F}\right\}_{z,0}-\mathrm{R}h=0. \quad \ldots \quad (17.)$$

From these, by very simple treatment, the form of F may be found; and from that form every required expression will be deduced with great facility.

14. A slight familiarity with the expressions for strains, as given by simple theory in some ordinary cases, is sufficient to convince us that F will contain only integer

powers of x and y. Assume, therefore,

$$F=S+Ty+Uy^2+Vy^3+Wy^4+\&c.,$$

where S, T, U, V, W, &c. are functions of x; then

$$\frac{dF}{dy}= \quad T+2Uy+3Vy^2+4Wy^3+\&c.$$

For r, s, the value of this is

$$T_r+2U_r.s+3V_r.s^2+4W_r.s^3+\&c.$$

For z, 0, its value is

$$T_z.$$

The expression $\left(\frac{dF}{dy}\right)_{r,s}-\left(\frac{dF}{dy}\right)_{z,0}$ will therefore contain the function T_z, where z is absolutely arbitrary. It is impossible that equation (15.) can subsist, except by making $T_z=0$, and therefore $T_r=0$, and generally $T=0$.

Again, omitting T, we find (using the accents to indicate differential coefficients)

$$\frac{dF}{dx}=S'+U'y^2+V'y^3+W'y^4+\&c.$$

For r, s, the value of this is

$$S'_r+U'_r.s^2+V'_r.s^3+W'_r.s^4+\&c.$$

For z, 0, its value is

$$S'_z.$$

For the same reason as before, S' generally $=0$. Therefore if S have any value, it is a mere numerical constant; and this will disappear in each of the equations (15.), (16.), (17.); and therefore it may be entirely omitted. The expression for F will therefore be reduced to $Uy^2+Vy^3+Wy^4+\&c.$ We shall hereafter show that ordinary investigations entitle us to assume that the expression for F will really be limited to the first two terms of this series, and that the powers of x will not be higher than the second; and therefore we shall suppose

$$F=(ax^2+bx+c)y^2+(ex^2+fx+g)y^3.$$

We can now proceed with instances.

15. *Example* 1. Suppose the beam to project from a wall, and to sustain no load except its own weight.

Here R=0; and the three equations (15.), (16.), (17.), with the last assumption for F, become

$$\begin{aligned}
&(2ar^2+2br+2c)s+(3er^2+3fr+3g)s^2=0,\\
&(2ar\ +b)s^2 \qquad +(2er\ +f)s^3 \qquad =0,\\
&(3ar^2+2br+c)s^2+(4er^2+3fr+2g)s^3=0.
\end{aligned}$$

Determining from these the values of b, c, g, we change the expression for F to the following:

$$F=\begin{Bmatrix}\{ax^2+(-2ar-2ers-fs)x+ar^2+2er^2s+frs\}y^2\\ +\{ex^2+fx-er^2-fr\}y^3\end{Bmatrix}.$$

To determine the constants a, e, f, which remain, we must have recourse to other considerations.

16. If we suppose the beam cut through in a vertical line corresponding to abscissa x, and if we make the usual assumptions in regard to the horizontal forces acting between the two parts and thus sustaining the moment of the distant part, namely, that there is a neutral point in the centre of the depth—that on the upper side of this neutral point the forces are forces of tension, and on the lower side are forces of compression—and that these forces are proportional to the distances from the neutral point, with equal coefficients on both sides,—then we can ascertain the horizontal force at every point. But I remark that it appears to me that these suppositions involve a distinct hypothesis as to the physical structure of the material. They seem to imply that the actual extensions or compressions correspond exactly to the curvature of the edge of the lamina, and that the forces of elasticity so put into play correspond to the amount of extension or compression. The experiments of Mr. W. H. BARLOW appear to modify this theory; and it seems probable that, when duly followed into their mathematical consequences, they may require the introduction into the formula for F of other powers of y. Leaving this question open, I shall now proceed, on the usual assumptions, to compute the horizontal force at every point of the vertical division.

17. Let the horizontal force at elevation y, estimated as compression, be represented by $c.\left(\frac{s}{2}-\ \right)$; the force on the element δy is the ribbon $\delta y\times c.\left(\frac{s}{2}-y\right)$; its moment is $y\times\delta y\times c.\left(\frac{s}{2}-y\right)=c\left(\frac{sy}{2}-y^2\right)\delta y$; and the entire moment is $c\int dy\left(\frac{sy}{2}-y^2\right)=c\left(\frac{sy^2}{4}-\frac{y^3}{3}\right)$; which, from $y=0$ to $y=s$, is $-\frac{cs^3}{12}$. The moment produced by the weight of the distant part of the bar is the product of its weight by the horizontal distance of its centre of gravity, or is $(r-x)\times s\times\frac{r-x}{2}=\frac{s(r-x)^2}{2}$. The equation of moments is therefore, $-\frac{cs^3}{12}+\frac{s(r-x)^2}{2}=0$. From this, $c=\frac{6(r-x)^2}{s^2}$; and the horizontal compression-force at elevation $y=\frac{6}{s^2}.(r-x)^2.\left(\frac{s}{2}-y\right)$; or the horizontal compression-force on the element $\delta y=\frac{6}{s^2}.(r-x)^2.\left(\frac{s}{2}-y\right)\delta y$.

18. But we have the means of expressing the same horizontal force in terms of F. For, in the last expressions of art. 7, conceive the dividing line to be vertical; that is, conceive $\delta x=0$, and $\delta s=\delta y$; then we have for the compression-force on the element δy in direction x, the expression $\mathrm{L}.\delta y$; which, giving to L its value from the end of art. 11, becomes $\frac{d^2\mathrm{F}}{dy^2}\delta y$.

Comparing these two expressions, $\frac{d^2\mathrm{F}}{dy^2}=\frac{6}{s^2}(r-x)^2.\left(\frac{s}{2}-y\right)$. And, using the last for-

mula of art. 14,

$$\left.\begin{array}{l}\{2ax^2+(-4ar-4ers-2fs)x+2ar^2+4er^2s+2frs\}\\+\{6ex^2+6fx-6er^2-6fr\}y\end{array}\right\}=\frac{6}{s^2}(r-x)^2.\left(\frac{s}{2}-y\right).$$

Comparing the coefficients of y, $ex^2+fx+(-er^2-fr)=-\frac{1}{s^2}x^2+\frac{2r}{s^2}x-\frac{r^2}{s^2}$. The first term gives $e=-\frac{1}{s^2}$; the second gives $f=\frac{2r}{s^2}$; the third gives $\frac{r^2}{s^2}-\frac{2r^2}{s^2}=-\frac{r^2}{s^2}$, which is identical.

Then substituting these in the term independent of y, and comparing,

$$2ax^2+\left(-4ar+\frac{4r}{s}-\frac{4r}{s}\right)x+2ar^2-\frac{4r^2}{s}+\frac{4r^2}{s}=\frac{3}{s}x^2-\frac{6r}{s}x+\frac{3}{s}r^2.$$

The first term gives $a=\frac{3}{2s}$; and this makes the second and third comparisons to become identical equations. The circumstance, that the determination of the constants from some terms causes the other terms to agree, gives evidence of the agreement of the two lines of theory, inasmuch as those remaining terms are obtained in the two theories by totally different operations, each peculiar to its own theory.

We may now therefore use $\frac{d^2\mathrm{F}}{dy^2}=\frac{6}{s^2}(r-x)^2.\left(\frac{s}{2}-y\right)$.

19. From this we find

$$\mathrm{F}=\frac{6}{s^2}.(r-x)^2.\left(\frac{sy^2}{4}-\frac{y^3}{6}\right);$$

from which

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2}=\frac{3}{s^2}.(r-x)^2.(s-2y)\quad=\quad s.\frac{3r^2}{s^2}.\left(1-\frac{x}{r}\right)^2.\left(1-\frac{2y}{s}\right);$$

$$\mathrm{M}=\frac{d^2\mathrm{F}}{dxdy}=\frac{-12}{s^2}.(r-x).\left(\frac{sy}{2}-\frac{y^2}{2}\right)=-s.\frac{6r}{s}.\left(1-\frac{x}{r}\right).\frac{y}{s}.\left(1-\frac{y}{s}\right);$$

$$\mathrm{N}=-2\mathrm{M}\quad=\quad s.\frac{12r}{s}.\left(1-\frac{x}{r}\right).\frac{y}{s}.\left(1-\frac{y}{s}\right);$$

$$\mathrm{O}=\frac{d^2\mathrm{F}}{dx^2}=\frac{12}{s^2}.\left(\frac{sy^2}{4}-\frac{y^3}{6}\right)\quad=\quad s.\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$\mathrm{Q}=y-\mathrm{O}=s\frac{y}{s}-\mathrm{O}=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)=s.\frac{y}{s}.\left(1-\frac{y}{s}\right).\left(1-\frac{2y}{s}\right).$$

Put $v=\frac{x}{r}$, $w=\frac{y}{r}$, and omit the general multiplier s. And as the succeeding operations, while kept in the symbolical form, become rather cumbrous, assume for $\frac{r}{s}$ a numerical value, as 5. Then

$$\mathrm{L}=75.(1-v)^2.(1-2w);$$

$$\mathrm{N}=60.(1-v).w.(1-w);$$

$$\mathrm{Q}=w.(1-w).(1-2w).$$

From these (see art. 7),

$$\tan 2\beta=\frac{N}{L+Q};$$

$$C-B=\frac{N}{\sin 2\beta}=\frac{L+Q}{\cos 2\beta};$$

$$C+B=L-Q;$$

which give the numerical values of the three elements B, C, β of the strains at every point.

By means of these formulæ, the numbers of Table I. (end of the Memoir) have been computed and the lines of pressure traced in Plate V. fig. 4. They give complete information on the nature and magnitude of the strains to which such a beam is subject.

20. *Example* 2. A beam of length $2r$ and depth s rests, at its two ends, freely on piers, and sustains no load except its own weight.

Assume, as before,

$$F=(ax^2+bx+c)y^2+(ex^2+fx+g)y^3,$$

and remark that the distant pier exerts a reaction vertically upwards, of magnitude rs at distance $2r$. The three equations (15.), (16.), (17.), taking the integrals from z, 0 to $2r$, s, become

$$\begin{aligned}
&(\ 8ar^2+4br+2c)s+(12er^2+6fr+3g)s^2 &&=0;\\
&(\ 4ar+b)s^2+(\ 4er+f)s^3-rs &&=0;\\
&(12ar^3+4br+c)s^2+(16er^2+6fr+2g)s^3-2r^2s &&=0.
\end{aligned}$$

When from these we determine the values of b, c, g, and substitute them in the expression for F, it becomes

$$F=\left\{\begin{array}{l}\left\{ax^2+\left(-4ar-4ers-fs+\frac{r}{s}\right)x+\left(4ar^2+8er^2s+2frs-\frac{2r^2}{s}\right)\right\}y^2\\ +\{ex^2+fx+(-4er^2-2fr)\}y^3\end{array}\right\}.$$

21. The horizontal pressure at any point of any vertical line across the beam at distance x will be found on the usual theory as follows. The compression at any elevation y being represented, as in article 17, by $c.\left(\frac{s}{2}-y\right)$, the entire moment is, as in that article, $-\frac{cs^3}{12}$. The moment produced by the weight of the distant part of the beam, whose length is $2r-x$, is $\frac{s(2r-x)^2}{2}$; and the moment produced by the reaction at the distant pier is $-rs\times(2r-x)$. The equation of moment is therefore

$$-\frac{cs^3}{12}+\frac{s(2r-x)^2}{2}-rs\times(2r-x)=0,$$

or

$$-\frac{cs^2}{12}+\frac{(2r-x)^2}{2}-\frac{2r(2r-x)}{2}=0,$$

or

$$-\frac{cs^2}{12}-\frac{x(2r-x)}{2}=0.$$

From this, $c=\frac{6x^2-12rx}{s^2}$; and the horizontal compression-force at elevation y

$$=\frac{6x^2-12rx}{s^2}\cdot\left(\frac{s}{2}-y\right).$$

Therefore, as in article 18,

$$\frac{d^2F}{dy^2}=\frac{6x^2-12rx}{s^2}\left(\frac{s}{2}-y\right).$$

And, using the last formula of article 20,

$$\left\{\begin{array}{l}\left\{2ax^2+\left(-8ar-8ers-2fs+\frac{2r}{s}\right)x+\left(8ar^2+16er^2s+4frs-\frac{4r^2}{s}\right)\right\}\\+\{6ex^2+6fx+(-24er^2-12fr)\}y\end{array}\right\}=\frac{6x^2-12rx}{s^2}\left(\frac{s}{2}-y\right).$$

Comparing the coefficients of y, $6ex^2+6fx+(-24er^2-12fr)=-\frac{6}{s^2}x^2+\frac{12r}{s^2}x$. The first term gives $e=-\frac{1}{s^2}$; the second gives $f=\frac{2r}{s^2}$; the third gives $\frac{24r^2}{s^2}-\frac{24r^2}{s^2}=0$, which is identical.

Substituting these in the term independent of y, and comparing,

$$2ax^2+\left(-8ar+\frac{8r}{s}-\frac{4r}{s}+\frac{2r}{s}\right)x+\left(8ar^2-\frac{16r^2}{s}+\frac{8r^2}{s}-\frac{4r^2}{s}\right)=\frac{3}{s}x^2-\frac{6r}{s}x.$$

The first term gives $a=\frac{3}{2s}$; and on substituting this, the second and third comparisons become identical equations. The evidence of correctness of theory is therefore satisfactory; and we may use $\frac{d^2F}{dy^2}=\frac{6x^2-12rx}{s^2}\left(\frac{s}{2}-y\right)$.

22. From this we find

$$F=\frac{6x^2-12rx}{s^2}\left(\frac{sy^2}{4}-\frac{y^3}{6}\right);$$

from which

$$L=\frac{d^2F}{dy^2}=\frac{6x^2-12rx}{s^2}\cdot\left(\frac{s}{2}-y\right)\qquad=-s\cdot\frac{6r^2}{s^2}\cdot\frac{x}{r}\cdot\left(1-\frac{x}{2r}\right)\cdot\left(1-\frac{2y}{s}\right);$$

$$M=\frac{d^2F}{dxdy}=\frac{12x-12r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\qquad=-s\cdot\frac{6r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$N=\quad -2M\qquad= s\cdot\frac{12r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$O=\frac{d^2F}{dx^2}=\frac{12}{s^2}\left(\frac{sy^2}{4}-\frac{y^3}{6}\right)\qquad= s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$Q=y-O=s\frac{y}{s}-O=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)= s\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right)\cdot\left(1-\frac{2y}{s}\right).$$

As before, put $v=\frac{x}{r}$, $w=\frac{y}{s}$; and suppose $\frac{r}{s}=5$. Then, omitting s,

$$L=-75\,.\,v\,.\,(2-v)\,.\,(1-2w);$$

$$N=\ 60\,.\,(1-v)\,.\,w\,.\,(1-w);$$

$$Q=\ w\,.\,(1-w)\,.\,(1-2w);$$

after which we may use the same formulæ as before, namely,

$$\tan 2\beta=\frac{N}{L+Q},$$

$$C-B=\frac{N}{\sin 2\beta}=\frac{L+Q}{\cos 2\beta},$$

$$C+B=L-Q;$$

by means of which the numbers have been computed for Table II. (end of the Memoir), and the lines have been traced that are exhibited in Plate VI. fig. 5.

23. There is one part of the pressures which it is matter of great interest to compute, namely, the pressures exerted on different parts of the end portion of the beam which rests on the pier. It will be seen in figure 7 that this part is not free from pressure; there are at every point a large force of compression in one direction, and a large force of tension in another direction. And the circumstances of this part differ from those of any other vertical section of the beam in this respect, that there is no opposing force. In all other sections, a thrust of compression on one side is met by a thrust of compression on the other side, and so for tension; and though there may be a tendency to crush or to disrupt the particles of the metal, yet there is no great tendency to force a small sectional portion horizontally or vertically. But on the end portion, where the forces of compression and tension are not so met, there are or may be great tendencies to force that end portion horizontally or vertically. We proceed now to investigate these tendencies.

24. First, for the horizontal pressure. The force B (which is estimated as a compression), acting in a direction which makes the angle β with the vertical, upon the element δy (as measured in the vertical direction) or $\sin\beta\,.\,\delta y$ (as measured in the direction transverse to B), does really exert the pressure $B\sin\beta\,.\,\delta y$ in the direction of B, or the pressure $B\,.\sin\beta\,.\,\delta y\times\sin\beta$, or $B\,.\,\sin^2\beta\,.\,\delta y$, in the horizontal direction. Similarly, the force C exerts the pressure $C\,.\,\sin^2(\beta+90°)\,.\,\delta y$, or $C\,.\,\cos^2\beta\,.\,\delta y$, in the horizontal direction. The entire horizontal force upon the element δy is therefore

$$(B\,.\,\sin^2\beta+C\,.\,\cos^2\beta)\,.\,\delta y=L\,.\,\delta y=\frac{d^2F}{dy^2}\delta y.$$

In the instance before us, of a beam resting on two piers,

$$\frac{d^2F}{dy^2}=\frac{6x^2-12rx}{s^2}\cdot\left(\frac{s}{2}-y\right);$$

and at the end of the beam, where $x=2r$, this quantity $=0$ whatever be the value of y. The same applies where $x=0$. There is no tendency therefore to bend or distort the end portion.

25. Secondly, for the vertical pressure. The pressure $B\,.\sin\beta\,.\,\delta y$ in the direction of B, found in last article, will produce the pressure $B\,.\sin\beta\,.\,\cos\beta\,.\,\delta y$ in the direction vertically downwards. Similarly, the force C will produce the pressure

$$C\,.\,\sin(\beta+90°)\,.\,\cos(\beta+90°)\,.\,\delta y$$

vertically downwards. The whole downwards pressure therefore on the element δy is $(\mathrm{B}.\sin\beta.\cos\beta+\mathrm{C}.\sin\overline{90^\circ+\beta}.\cos\overline{90^\circ+\beta}).\delta y$, or $(\mathrm{B}-\mathrm{C}).\sin\beta.\cos\beta.\delta y$, or $\mathrm{M}.\delta y$; which in the present instance $=\frac{12x-12r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\cdot\delta y$. At the end of the beam, where $x=2r$, this $=\frac{6r}{s^2}.(sy-y^2).\delta y$. Let $y'=s-y$ (that is, let the ordinate be measured from the upper edge downwards); then the downwards pressure on the element $\delta y'$ of the end portion $=\frac{6r}{s^2}\cdot(sy'-y'^2).\delta y'$. Integrating this from the top downwards, we find for the pressure which a horizontal section of the end portion must sustain,

$$\frac{6r}{s^2}\cdot\left(\frac{sy'^2}{2}-\frac{y'^3}{3}\right)=rs\cdot\frac{y'^2}{s^2}\left(3-\frac{2y'}{s}\right).$$

At the middle of the depth this $=\frac{rs}{2}$; at the base it $=rs$. It appears therefore that every part of the end portion which rests upon the pier is subject to a very heavy pressure (such as affects no other part of the beam), increasing from the top to the bottom, where it is equal to the weight of half the beam.

It was undoubtedly from a clear perception of the magnitude of this pressure (though not reduced to the formulæ of mathematical investigation) that Mr. ROBERT STEPHENSON, in the construction of the Britannia Bridge, was induced to insert the strong end-frames in each of the tubes, at the places where they rest on their piers.

26. *Example* 3. A beam of length $2r$ and depth s rests, at its two ends, freely on piers, and carries a weight W at the distance a from the left-hand extremity.

For convenience, we will suppose a to be not greater than r. This will include every case, as the supposition a' greater than r is the same as the supposition a less than r measured from the right-hand extremity, if $a+a'=2r$.

In examples 1 and 2, we have selected a form for F which satisfied the equations (15.), (16.), (17.), applying to F, and we have then shown that this form represents properly the horizontal pressure determined from the ordinary theory. In the present example, which is unsymmetrical and complicated, we shall find the form for F (a discontinuous form) which represents the horizontal pressure as determined from the ordinary theory, and shall show that this form satisfies in all parts the equations (15.), (16.), (17.).

27. The pressure upon the left-hand pier is $rs+\mathrm{W}\cdot\frac{2r-a}{2r}$; and that upon the right-hand pier is $rs+\mathrm{W}\cdot\frac{a}{2r}$. The reactions of the piers have the same values, but in the opposite direction. For a transverse section at the ordinate x, where x is less than a, the forces which produce moments are the following: the weight $s\times(2r-x)$ acting at distance $\frac{2r-x}{2}$; the weight W at distance $a-x$; and the reaction $rs+\mathrm{W}\cdot\frac{a}{2r}$ at distance $2r-x$. The sum of their moments, estimated as compressing the upper part, is

$$(2r-x).\left(rs+\mathrm{W}\cdot\frac{a}{2r}-s\cdot\frac{2r-x}{2}\right)-\mathrm{W}(a-x),\ =\left(rs+\mathrm{W}\cdot\frac{2r-a}{2r}\right)x-\frac{s}{2}x^2.$$

(The same value will be found if we consider the moment as produced by the weight of bar and the reaction on the left side of x.) Treating this as in article 17, we find the horizontal compression-force at elevation y

$$=\frac{6}{s^2}\left\{\left(2r+\mathrm{W}\frac{2r-a}{rs}\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right).$$

This, as in preceding instances, ought to equal $\frac{d^2\mathrm{F}}{dy^2}$; and therefore F ought to equal

$$\frac{6}{s^2}\left\{\left(2r+\mathrm{W}\cdot\frac{2r-a}{rs}\right)x-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right).$$

This formula applies to any point of the part of the bar included between $x=0$ and $x=a$, which we shall call the "first part." For any point of the "second part," or the part included between $x=a$ and $x=2r$, there is no weight W on the right hand; the forces producing moments are the weight $s\times(2r-x)$ acting at distance $\frac{2r-x}{2}$, and the reaction $rs+\mathrm{W}\frac{a}{2r}$ at distance $2r-x$; the sum of their moments, estimated as compressing the upper part, is

$$(2r-x)\left(rs+\mathrm{W}\frac{a}{2r}-s\frac{2r-x}{2}\right)=\mathrm{W}a+\left(rs-\mathrm{W}\frac{a}{2r}\right)x-\frac{s}{2}x^2;$$

whence, as in article 17, the horizontal compression-force at elevation y

$$=\frac{6}{s^2}\left\{\frac{2\mathrm{W}a}{s}+\left(2r-\mathrm{W}\frac{a}{rs}\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right),$$

which ought to equal $\frac{d^2\mathrm{F}}{dy^2}$; and therefore F ought to equal

$$\frac{6}{s^2}\left\{\frac{2\mathrm{W}a}{s}+\left(2r-\mathrm{W}\cdot\frac{a}{rs}\right)x-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right).$$

This formula applies to any point of the part of the bar included between $x=a$ and $x=2r$, or to any point of the "second part." The function changes its form, or is discontinuous, when x passes the value a,—the two formulæ, however, giving the same value for F when $x=a$. We have now to ascertain whether the discontinuous function does in all parts satisfy the equations (15.), (16.), (17.).

28. First, suppose the integrals to begin from a point z in the "first part." It is unnecessary to make an elaborate trial of equation (15.), because, as our assumed value for F contains the multiplier $\frac{y^3}{6}-\frac{sy^2}{4}$, and $\frac{d\mathrm{F}}{dy}$ therefore contains the multiplier $\frac{y^2}{2}-\frac{sy}{2}$, $\frac{d\mathrm{F}}{dy}$ will necessarily vanish at both the limits for y (namely $y=0$, $y=s$) which enter into the formulæ of (15.). In regard to the other equations, the integrals must be taken by the formulæ of the "first part" from z, 0, to a, s; and by the formulæ of the "second part" from a, s, to $2r$, s; and the constant forces are $+\mathrm{W}$ at abscissa a and $-\left(rs+\mathrm{W}\frac{a}{2r}\right)$ at abscissa $2r$.

For equation (16.), $\frac{d\mathrm{F}}{dx}$ in the "first part" $=\frac{6}{s^2}\left\{2r+\mathrm{W}\,.\,\frac{2r-a}{rs}-2x\right\}\left(\frac{y^3}{6}-\frac{sy^2}{4}\right)$; which for z, 0, $=0$, and for a, s, $=\frac{-6}{s^2}\left\{2r-2a+\mathrm{W}\,.\,\frac{2r-a}{rs}\right\}\cdot\frac{s^3}{12}$. And $\frac{d\mathrm{F}}{dx}$ in the "second part"

$$=\frac{6}{s^2}\left\{2r-\mathrm{W}\frac{a}{rs}-2x\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right);$$

which for a, s,

$$=\frac{-6}{s^2}\left\{2r-2a-\mathrm{W}\frac{a}{rs}\right\}\cdot\frac{s^3}{12},$$

and for $2r$, s,

$$=-\frac{6}{s^2}\left\{-2r-\mathrm{W}\frac{a}{rs}\right\}\frac{s^3}{12}.$$

The sum of the two portions of the integral will therefore be

$$-\frac{s}{2}\left\{0+2r-2a+\mathrm{W}\,.\,\frac{2r-a}{rs}-2r+2a+\mathrm{W}\frac{a}{rs}-2r-\mathrm{W}\frac{a}{rs}\right\}=-\frac{s}{2}\left\{-2r+\mathrm{W}\frac{2r-a}{rs}\right\}$$
$$=rs-\mathrm{W}\cdot\frac{2r-a}{2r}.$$

To this are to be added $+\mathrm{W}$ and $-\left(rs+\mathrm{W}\frac{a}{2r}\right)$, or $-rs+\mathrm{W}\frac{2r-a}{2r}$; the sum is 0. Equation (16.) therefore is satisfied when z is in the "first part."

For equation (17.): omitting $y\,\frac{d\mathrm{F}}{dy}$ (because, as is explained above, it cannot produce any term), it will be found that in the "first part"

$$x\frac{d\mathrm{F}}{dx}-\mathrm{F}=\frac{6}{s^2}\cdot\left\{-x^2\right\}\cdot\left(\frac{y}{6}-\frac{sy^2}{4}\right);$$

which for z, 0, $=0$, and for a, s, $=\frac{6}{s^2}\cdot a^2\cdot\frac{s^3}{12}$. And in the "second part,"

$$x\frac{d\mathrm{F}}{dx}-\mathrm{F}=\frac{6}{s^2}\left\{-\frac{2\mathrm{W}a}{s}-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right);$$

which for a, s,

$$=\frac{6}{s^2}\left\{\frac{2\mathrm{W}a}{s}+a^2\right\}\frac{s^3}{12},$$

and for $2r$, s,

$$=\frac{6}{s^2}\left\{\frac{2\mathrm{W}a}{s}+4r^2\right\}\cdot\frac{s^3}{12}.$$

The sum of the two portions of the integral will therefore be

$$\frac{s}{2}\left\{0+a^2-\frac{2\mathrm{W}a}{s}-a^2+\frac{2\mathrm{W}a}{s}+4r^2\right\}=2r^2s.$$

To this are to be added $+\mathrm{W}a$ and $-\left(rs+\mathrm{W}\frac{a}{2r}\right)2r$, or $-2r^2s$; the sum is 0. Equation (17.) therefore is satisfied when z is in the "first part."

29. Second, suppose the integrals to begin from a point z in the "second part." As before, it is unnecessary to make a trial of equation (15.), which is necessarily satisfied. In regard to equations (16.) and (17.), the integrals are only to be taken by the formulæ

of the "second part" from z, 0, to $2r$, s; and the only constant force is $-\left(rs+\mathrm{W}\frac{a}{2r}\right)$ at abscissa $2r$.

For equation (16.), $\frac{d\mathrm{F}}{dx}$ in the "second part" $=\frac{6}{s^2}\left\{2r-\mathrm{W}\frac{a}{rs}-2x\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right)$, which for z, 0, $=0$, and for $2r$, s, $=-\frac{s}{2}\left\{-2r-\mathrm{W}\frac{a}{rs}\right\}$, or $rs+\mathrm{W}\frac{a}{2r}$. To this is to be added $-\left(rs+\mathrm{W}\frac{a}{2r}\right)$; the sum is 0. Equation (16.) therefore is satisfied when z is in the "second part."

For equation (17.), $x\frac{d\mathrm{F}}{dx}-\mathrm{F}$ in the "second part" $=\frac{6}{s^2}\left\{-\frac{2\mathrm{W}a}{s}-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right)$, which for z, 0, $=0$, and for $2r$, s, $=\frac{s}{2}\left\{\frac{2\mathrm{W}a}{s}+4r^2\right\}=+\mathrm{W}a+2r^2s$. To this is to be added $-\left(rs+\mathrm{W}\frac{a}{2r}\right)2r$, or $-2r^2s-\mathrm{W}a$; the sum is 0. Equation (17.) therefore is satisfied when z is in the "second part."

30. It appears therefore that our equations (15.), (16.), (17.) are in all parts of this loaded bar satisfied by the discontinuous formula which we found for F; and therefore that formula is to be adopted in the further calculations. But different calculations must be made for the "first part" and the "second part."

First Part, from $x=0$ to $x=a$.

$$\mathrm{F}=\frac{6}{s^2}\cdot\left\{\left(2r+\mathrm{W}\cdot\frac{2r-a}{rs}\right)x-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right);$$

$$\mathrm{L}=\frac{6}{s^2}\cdot\left\{\left(2r+\mathrm{W}\cdot\frac{2r-a}{rs}\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right);$$

$$\mathrm{M}=\frac{6}{s^2}\cdot\left\{2r+\mathrm{W}\cdot\frac{2r-a}{rs}-2x\right\}\cdot\left(\frac{y^2}{2}-\frac{sy}{2}\right);$$

$$\mathrm{N}=\frac{6}{s^2}\cdot\left\{2r+\mathrm{W}\cdot\frac{2r-a}{rs}-2x\right\}\cdot(sy-y^2);$$

$$\mathrm{O}=s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$\mathrm{Q}=s\cdot\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right).$$

Second Part, from $x=a$ to $x=2r$.

$$\mathrm{F}=\frac{6}{s^2}\cdot\left\{\frac{2\mathrm{W}a}{s}+\left(2r-\mathrm{W}\cdot\frac{a}{rs}\right)x-x^2\right\}\cdot\left(\frac{y}{6}-\frac{sy^2}{4}\right);$$

$$\mathrm{L}=\frac{6}{s^2}\cdot\left\{\frac{2\mathrm{W}a}{s}+\left(2r-\mathrm{W}\cdot\frac{a}{rs}\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right);$$

$$\mathrm{M}=\frac{6}{s^2}\left\{2r-\mathrm{W}\cdot\frac{a}{rs}-2x\right\}\cdot\left(\frac{y^2}{2}-\frac{sy}{2}\right);$$

$$\mathrm{N}=\frac{6}{s^2}\left\{2r-\mathrm{W}\frac{a}{rs}-2x\right\}\cdot(sy-y^2);$$

$$\mathrm{O}=s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$\mathrm{Q}=s\cdot\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right).$$

To diminish the number of symbols, we will at once assume that W= weight of half the bar $=rs$. Then we have

$$\mathrm{L}=\frac{6}{s^2}\cdot\left\{(4r-a)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right);$$

$$\mathrm{N}=\frac{6}{s^2}\cdot\left\{4r-a-2x\right\}\cdot(sy-y^2);$$

$$\mathrm{Q}=s\cdot\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right).$$

$$\mathrm{L}=\frac{6}{s^2}\left\{2ra+(2r-a)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right);$$

$$\mathrm{N}=\frac{6}{s^2}\left\{2r-a-2x\right\}\cdot(sy-y^2);$$

$$\mathrm{Q}=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right).$$

And we will now select the cases which it appears desirable to compute numerically.

31. The strains upon the beam are not at all affected by placing a weight upon its end (supposed strong enough to resist distortion of form). It appears probable, therefore, that the extreme changes of opposite character will be given, on the one hand, by placing the weight upon the centre of the beam's length, or making $a=r$; on the other hand by placing the weight upon the centre of one half of the beam, or making $a=\frac{r}{2}$. We will proceed first with the formulæ for the case when the weight is upon the centre, or $a=r$. It is unnecessary here to make calculations for the two segments of the beam, as the strains will be symmetrical with respect to the two extremities. As before, $\frac{r}{s}$ is taken $=5$.

Weight rs placed on the centre of the beam's length.

$$\mathrm{L}=\frac{-6}{s^2}\left\{3rx-x^2\right\}.\left(\frac{s}{2}-y\right)=-s.\frac{3r^2}{s^2}.\left\{\frac{3x}{r}-\frac{x^2}{r^2}\right\}.\left(1-\frac{2y}{s}\right)=-s.75.v.(3-v).(1-2w);$$

$$\mathrm{N}=\frac{6}{s^2}\{3r-2x\}.y.(s-y)=s.\frac{12r}{s}.\left\{\frac{3}{2}-\frac{x}{r}\right\}.\frac{y}{s}\left(1-\frac{y}{s}\right)=s.60.\left\{\frac{3}{2}-v\right\}.w.(1-w);$$

$$\mathrm{Q}\qquad=s.w.(1-w).(1-2w);$$

from all which, as before, the general factor s may be omitted.

Proceeding now with the other case, or

Weight rs placed on the centre of the first half of the beam's length,

the formulæ for the "first part," from $x=0$ to $x=a=\frac{r}{2}$, or from $v=0$ to $v=0{\cdot}5$, will be

$$\mathrm{L}=\frac{-6}{s^2}.\left\{\frac{7r}{2}x-x^2\right\}.\left(\frac{s}{2}-y\right)=-s.\frac{3r^2}{s^2}.\left\{\frac{7}{2}.\frac{x}{r}-\frac{x^2}{r^2}\right\}.\left(1-\frac{2y}{s}\right)=-s.75.v\left(\frac{7}{2}-v\right).(1-2w);$$

$$\mathrm{N}=\frac{6}{s^2}.\left\{\frac{7r}{2}-2x\right\}y.(s-y)=s.\frac{12r}{s}.\left\{\frac{7}{4}-\frac{x}{r}\right\}.\frac{y}{s}.\left(1-\frac{y}{s}\right)=s.60.\left\{\frac{7}{4}-v\right\}.w.(1-w);$$

$$\mathrm{Q}\qquad=s.w.(1-w).(1-2w);$$

and those for the "second part," from $x=\frac{r}{2}$ to $x=2r$, or from $v=0{\cdot}5$ to $v=2{\cdot}0$, will be

$$\mathrm{L}=-\frac{6}{s^2}.\left\{r^2+\frac{3r}{2}x-x^2\right\}.\left\{\frac{s}{2}-y\right\}=-s.\frac{3r^2}{s^2}.\left(2-\frac{x}{r}\right).\left(\frac{1}{2}+\frac{x}{r}\right).\left(1-\frac{2y}{s}\right)=-s.75.(2-v).\left(\frac{1}{2}+v\right).(1-2w);$$

$$\mathrm{N}=\frac{6}{s^2}.\left\{\frac{3}{2}-2x\right\}.(sy-y^2)=s.\frac{12r}{s}.\left\{\frac{3}{4}-\frac{x}{r}\right\}.\frac{y}{s}.\left(1-\frac{y}{s}\right)=s.60.\left(\frac{3}{4}-v\right).w.(1-w);$$

$$\mathrm{Q}\qquad=s.w.(1-w).(1-2w);$$

from all which the factor s may be omitted.

32. For all these cases, the same formulæ as before are to be used in the ultimate

calculations of the magnitudes and directions of the strains, namely,

$$\tan 2\beta=\frac{N}{L+Q},$$

$$C-B=\frac{N}{\sin 2\beta}=\frac{L+Q}{\cos 2\beta},$$

$$C+B=L-Q.$$

By means of these, the numbers have been computed for Table III., and Table IV. parts 1 and 2 (end of the Memoir), and the lines of figs. 6 and 7, Plate VI. have been traced.

33. It is worthy of remark that, in figures 4 and 5, the lines representing the direction of thrust, and also those representing the direction of pull, are continuous; but in figures 6 and 7 they are discontinuous, the two segments of each curve, at their meeting in the ordinate vertically below the weight, having different tangential directions. This follows as an inevitable consequence of the assumption in art. 16; I think it probable that a hypothesis like that of Mr. W. H. Barlow would remove the discontinuity. An investigation similar to that of art. 25 would show that, at these points, the transverse section of the beam must be sufficiently strong to support the weight by thrust (if the weight is on the top of the beam), or by tension (if the weight is carried by or attached to the bottom of the beam).

34. There are cases somewhat different from those already considered, whose importance and singularity of principle are such as to make them worthy of special notice. In Mr. Robert Stephenson's construction of the Britannia Bridge, the strength of the tubes was nearly doubled by the following admirable arrangement. The junction of the ends of successive tubes, at their meeting on the piers, was effected, not while the two successive tubes rested on the bearings which they were finally to take, but while the distant end of one of the tubes was considerably elevated. It is a problem of no great difficulty to ascertain what elevation ought to be given in order to reduce the maximum strains on the bridge to their smallest value; when the best arrangement is made, the strains are reduced to one-half of their original value. The singularity of the mathematical principle consists in this, that there is impressed on the end-frame of the tube or beam a strain of the nature of a couple, or (as it is called in the preceding articles) a moment. Where there are three or more connected tubes, the middle tube, or each of the middle tubes, has such a moment-strain at each end; but each of the external tubes has a moment-strain at one end only (inasmuch as, at the land termination of the bridge, there are no means of applying such a strain). There are therefore two different cases, requiring different investigations.

35. Take, first, the case of a middle tube in which a moment-strain is impressed on each end, the directions of the two strains (supposed equal) being opposed, so that both tend to raise the middle of the tube. The pressures upon the two piers will not be disturbed, because the effects of the two strains upon the entire beam balance. If now we consider the forces which act on the distant part of the beam (using the lan-

uage of art. 7), we shall have to combine, with forces formerly recognized, the moment which acts on the distant end. By the known laws of translation of the place of application of a moment, we may suppose this moment applied at the imaginary division of the bar. Thus, at every vertical section of the bar, there is combined with the ordinary moment of strains a moment equal to that impressed on each end. The most advantageous magnitude for this moment is evidently half the magnitude of moment at the beam's centre, with opposite sign; for if we use a smaller value we leave too much moment at the centre, and if we use a larger value we impress too great a straining moment at the junction above the pier.

36. Now in art. 21 we found, for the horizontal thrust in a point of any vertical section, $\frac{6x^2-12rx}{s^2}\cdot\left(\frac{s}{2}-y\right)$. As regards the variations of x, this is greatest when $x=r$, and its value is then $-\frac{6r^2}{s^2}\left(\frac{s}{2}-y\right)$. One half of this with changed sign, or $+\frac{3r^2}{s^2}\left(\frac{s}{2}-y\right)$, is now to be applied to the expression for horizontal thrust in every part of the beam's length. Hence the expression to be used for horizontal thrust or compression is

$$\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{s}{2}-y\right),$$

and therefore

$$F=\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right).$$

It will be seen immediately that this quantity satisfies the equations (15.) and (16.), the integrals being taken from z, 0 to $2r$, s. But with regard to equation (17.), we must consider that in the instance before us a moment is to be introduced which has not presented itself before, namely, the moment impressed on the distant end. The value of that moment, which (with the sign contemplated in forming equation (17.)) is $-\int dy\cdot\frac{3r^2}{s^2}\left(\frac{sy}{2}-y^2\right)$, becomes $+\frac{r^2s}{4}$. Hence equation (17.) becomes in this case

$$\left\{y\frac{dF}{dy}+x\frac{dF}{dx}-F\right\}_{2r,\,s}-\left\{y\frac{dF}{dy}+x\frac{dF}{dx}-F\right\}_{z,\,0}-2r^2s+\frac{r^2s}{4}=0.$$

And, on making the substitutions, this equation is satisfied.

37. Therefore we are to adopt

$$F=\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right),$$

from which

$$L=\frac{d^2F}{dy^2}=\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{s}{2}-y\right)=\quad s\cdot\frac{3r^2}{s^2}\cdot\left(\frac{1}{2}-\frac{2x}{r}+\frac{x^2}{r^2}\right)\cdot\left(1-\frac{2y}{s}\right);$$

$$M=\frac{d^2F}{dxdy}=\frac{12x-12r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\quad=-s\cdot\frac{6r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$N=-2M\qquad\qquad=\quad s\cdot\frac{12r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$\mathrm{O}=\frac{d^2\mathrm{F}}{dx^2}=\frac{12}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right) \qquad =s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$\mathrm{Q}=y-\mathrm{O}=s\frac{y}{s}-\mathrm{O}=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)=s\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right)\cdot\left(1-\frac{2y}{s}\right).$$

Or, if $v=\frac{x}{r}$, $w=\frac{y}{s}$, $\frac{r}{s}=5$, and the multiplier s be omitted,

$$\mathrm{L}=75\,.\left\{(1-v)^2-\frac{1}{2}\right\}.(1-2w);$$

$$\mathrm{N}=60\,.(1-v)\,.\,w\,.(1-w);$$

$$\mathrm{Q}=w\,.(1-w).(1-2w).$$

Then

$$\tan 2\beta=\frac{\mathrm{N}}{\mathrm{L}+\mathrm{Q}};\quad \mathrm{C}-\mathrm{B}=\frac{\mathrm{N}}{\sin 2\beta}=\frac{\mathrm{L}+\mathrm{Q}}{\cos 2\beta};\quad \mathrm{C}+\mathrm{B}=\mathrm{L}-\mathrm{Q},$$

by which the numbers for Table V. have been computed, and the curves of figure 8, Plate VII. have been drawn.

38. Take, secondly, the case of an end tube, on which a moment is impressed only at one end. In this case, the effect of that moment is not balanced by a moment impressed at the other end, and must be balanced by an increase of pressure on the near pier (at which the moment is impressed), and a decrease of pressure on the distant pier. The value $\frac{r^2s}{4}$ of moment will be balanced by an increase of pressure $\frac{rs}{8}$ on the near pier, and a decrease of pressure $\frac{rs}{8}$ on the distant pier. Hence the pressure on the distant pier will be $rs-\frac{rs}{8}=\frac{7rs}{8}$. From this (as in art. 21) the moment produced by the weight of the distant part of the beam $=\frac{s(2r-x)^2}{2}$, and the moment produced by the reaction of the distant pier is $=-\frac{7rs}{8}\times(2r-x)$. The equation of moments is now

$$-\frac{cs^3}{12}+\frac{s(2r-x)^2}{2}-\frac{7rs}{8}\times(2r-x)=0;$$

or

$$-\frac{cs^3}{12}+s(2r-x).\left\{r-\frac{x}{2}-\frac{7r}{8}\right\}=0;$$

or

$$-\frac{cs^3}{12}-s(2r-x).\left(\frac{x}{2}-\frac{r}{8}\right)=0$$

From this,

$$c=\frac{(6x-12r)\left(x-\frac{r}{4}\right)}{s^2};$$

and the horizontal compression-force at elevation y

$$=\frac{(6x-12r)\left(x-\frac{r}{4}\right)}{s^2}\left(\frac{s}{2}-y\right)$$

Therefore we are to take for trial

$$F=\frac{(6x-12r)\left(x-\frac{r}{4}\right)}{s^2}\left(\frac{sy^2}{4}-\frac{y^3}{6}\right).$$

Remarking that the reaction of the distant pier $=\frac{7rs}{8}$, and that its moment upwards $=\frac{7rs}{8}\times 2r$, it will be found that this function satisfies equations (15.), (16.), (17.).

39. Adopting therefore

$$F=\frac{(6x-12r)\cdot\left(x-\frac{r}{4}\right)}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right),$$

we have

$$L=\frac{d^2F}{dy^2}=\frac{(6x-12r)\cdot\left(x-\frac{r}{4}\right)}{s^2}\cdot\left(\frac{s}{2}-y\right)=\quad s\cdot\frac{3r^2}{s^2}\cdot\left(\frac{x}{r}-2\right)\cdot\left(\frac{x}{r}-\frac{1}{4}\right)\cdot\left(1-\frac{2y}{s}\right);$$

$$M=\frac{d^2F}{dxdy}=\frac{12x-\frac{27}{2}r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\qquad=-s\cdot\frac{6r}{s}\cdot\left(\frac{9}{8}-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$N=-2M\qquad=\quad s\cdot\frac{12r}{s}\cdot\left(\frac{9}{8}-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$O=\frac{d^2F}{dx^2}=\frac{12}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right)\qquad=\quad s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$Q=y-O\qquad=\quad s\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right)\cdot\left(1-\frac{2y}{s}\right).$$

And, with $v=\frac{x}{r}$, $w=\frac{y}{s}$, $\frac{r}{s}=5$, $s=1$,

$$L=-75\,.(2-v).\left(v-\frac{1}{4}\right)\cdot(1-2w);$$

$$N=\quad 60\cdot\left(\frac{9}{8}-v\right).\,w\,.(1-w);$$

$$Q=\quad w\,.(1-w).(1-2w);$$

$$\tan 2\beta=\frac{N}{L+Q};\quad C-B=\frac{N}{\sin 2\beta}=\frac{L+Q}{\cos 2\beta};\quad C+B=L-Q;$$

by which the numbers for Table VI. have been computed, and the curves of figure 9 have been drawn.

40. These instances will probably suffice as applications of the theory to the most important cases of practice, and as examples of the modifications on subordinate points which may be required in investigating strains where the forms or other circumstances are different from those considered here.

41. Perhaps useful information may be derived from the diagrams and tables of numbers for guiding the construction of Latticed Bridges. Thus, in such cases as those of figures 5, 6, 7, the upper and lower edges require great longitudinal strength in the middle of the beam's length, but very little near the ends; on the contrary, powerful lattice-work is required near the ends, but very little near the middle. In the case of figure 8 these remarks require very considerable modification.

Table I.—Strains on the interior points of a beam which projects from a wall, supporting no other weight. The length of the beam is supposed to be five times its depth. The two numbers in each division of the Table are the values of the two principal strains, the unit being the depth of the beam. The positive sign denotes compression, and the negative sign tension. The angle is that by which the first-written strain is inclined to y, in the direction of diminishing x for increase of y. The direction of the second-written strain is at right angles to that of the first-written strain.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the wall, to the whole length of the beam).										
	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	−75·00 0·00 90° 0′	−60·75 0·00 90° 0′	−48·00 0·00 90° 0′	−36·75 0·00 90° 0′	−27·00 0·00 90° 0′	−18·75 0·00 90° 0′	−12·00 0·00 90° 0′	− 6·75 0·00 90° 0′	− 3·00 0·00 90° 0′	− 0·75 0·00 90° 0′	0·00 0·00
0·9	−60·12 + 0·19 87 30	−48·72 + 0·19 87 10	−38·52 + 0·19 86 50	−29·52 + 0·19 86 20	−21·72 + 0·19 85 40	−15·12 + 0·19 84 50	− 9·72 + 0·19 83 40	− 5·52 + 0·19 81 40	− 2·51 + 0·18 78 10	− 0·70 + 0·17 70 40	0·00 0·00
0·8	−45·51 + 0·60 84 0	−36·95 + 0·60 83 20	−29·30 + 0·60 82 30	−22·55 + 0·59 81 30	−16·69 + 0·59 80 20	−11·74 + 0·58 78 30	− 7·68 + 0·57 76 10	− 4·50 + 0·55 72 40	− 2·20 + 0·50 67 20	− 0·73 + 0·38 59 50	0·00 0·00
0·7	−31·27 + 1·35 78 40	−25·56 + 1·34 77 30	−20·44 + 1·32 76 10	−15·92 + 1·30 74 30	−11·98 + 1·27 72 40	− 8·64 + 1·22 70 10	− 5·87 + 1·15 67 0	− 3·66 + 1·04 63 10	− 1·97 + 0·86 58 30	− 0·77 + 0·55 53 30	0·00 0·00
0·6	−17·89 + 2·94 68 10	−14·95 + 2·85 66 40	−12·29 + 2·74 65 0	− 9·90 + 2·60 63 10	− 7·78 + 2·43 61 10	− 5·92 + 2·22 58 50	− 4·31 + 1·95 56 30	− 2·92 + 1·62 54 0	− 1·75 + 1·20 51 20	− 0·78 + 0·68 48 50	0·00 0·00
0·5	− 7·50 + 7·50 45 0	− 6·75 + 6·75 45 0	− 6·00 + 6·00 45 0	− 5·25 + 5·25 45 0	− 4·50 + 4·50 45 0	− 3·75 + 3·75 45 0	− 3·00 + 3·00 45 0	− 2·25 + 2·25 45 0	− 1·50 + 1·50 45 0	− 0·75 + 0·75 45 0	0·00 0·00
0·4	− 2·94 +17·89 21 50	− 2·85 +14·95 23 20	− 2·74 +12·29 25 0	− 2·60 + 9·90 26 50	− 2·43 + 7·78 28 50	− 2·22 + 5·92 31 10	− 1·95 + 4·31 33 30	− 1·62 + 2·92 36 0	− 1·20 + 1·75 38 40	− 0·68 + 0·78 41 10	0·00 0·00
0·3	− 1·35 +31·27 11 20	− 1·34 +25·56 12 30	− 1·32 +20·44 13 50	− 1·30 +15·92 15 30	− 1·27 +11·98 17 20	− 1·22 + 8·64 19 50	− 1·15 + 5·87 23 0	− 1·04 + 3·66 26 50	− 0·86 + 1·97 31 30	− 0·55 + 0·77 36 30	0·00 0·00
0·2	− 0·60 +45·51 6 0	− 0·60 +36·95 6 40	− 0·60 +29·30 7 30	− 0·59 +22·55 8 30	− 0·59 +16·69 9 40	− 0·58 +11·74 11 30	− 0·57 + 7·68 13 50	− 0·55 + 4·50 17 20	− 0·50 + 2·20 22 40	− 0·38 + 0·73 30 10	0·00 0·00
0·1	− 0·19 +60·12 2 30	− 0·19 +48·72 2 50	− 0·19 +38·52 3 10	− 0·19 +29·52 3 40	− 0·19 +21·72 4 20	− 0·19 +15·12 5 10	− 0·19 + 9·72 6 20	− 0·19 + 5·52 8 20	− 0·18 + 2·51 11 50	− 0·17 + 0·70 19 20	0·00 0·00
0·0	0·00 +75·00 0 0	0·00 +60·75 0 0	0·00 +48·00 0 0	0·00 +36·75 0 0	0·00 +27·00 0 0	0·00 +18·75 0 0	0·00 +12·00 0 0	0·00 + 6·75 0 0	0·00 + 3·00 0 0	0·00 + 0·75 0 0	0·00 0·00

TABLE II.—Strains on the interior points of a beam whose two ends rest upon piers, and which supports no other weight. The length of the beam is supposed to be ten times its depth. The two numbers in each division of the Table are the values of the two principal strains, the unit being the depth of the beam. The positive sign denotes compression, and the negative sign tension. The angle is that by which the first-written strain is inclined to y, in the direction of diminishing x for increase of y. The direction of the second-written strain is at right angles to that of the first-written strain.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from one pier, to half the length of the beam).										
	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	0·00 00·0 ° ′	0·00 +14·25 0° 0′	0·00 +27·00 0° 0′	0·00 +38·25 0° 0′	0·00 +48·00 0° 0′	0·00 +56·25 0° 0′	0·00 +63·00 0° 0′	0·00 +68·25 0° 0′	0·00 +72·00 0° 0′	0·00 +74·25 0° 0′	0·00 +75·00 0° 0′
0·9	− 2·66 + 2·74 45 20	− 0·43 +11·90 11 40	− 0·14 +21·82 5 40	− 0·05 +30·72 3 30	0·00 +38·47 2 30	+ 0·03 +45·04 1 40	+ 0·05 +50·42 1 10	+ 0·06 +54·61 0 50	+ 0·07 +57·61 0 30	+ 0·07 +59·40 0 20	+ 0·07 60·00 0 0
0·8	− 4·75 + 4·85 45 20	− 1·72 +10·37 22 50	− 0·77 +17·07 12 50	− 0·39 +23·43 8 10	− 0·19 +29·09 5 40	− 0·08 +33·92 4 0	0·00 +37·90 2 50	+ 0·05 +41·00 2 0	+ 0·08 +43·22 1 20	+ 0·09 +44·56 0 40	+ 0·10 +45·00 0 0
0·7	− 6·26 + 6·34 45 10	− 3·44 + 9·22 31 50	− 1·91 +12·80 21 40	− 1·10 +16·49 15 0	− 0·64 +19·92 10 50	− 0·35 +22·94 7 50	− 0·17 +25·45 5 40	− 0·05 +27·43 4 0	+ 0·03 +28·86 2 30	+ 0·07 +29·71 1 10	+ 0·08 +30·00 0 0
0·6	− 7·18 + 7·22 45 10	− 5·18 + 8·08 38 50	− 3·63 + 9·08 32 30	− 2·46 +10·16 26 30	− 1·62 +11·26 21 0	− 1·01 +12·31 16 20	− 0·59 +13·23 12 20	− 0·29 +13·99 8 50	− 0·10 +14·54 5 40	+ 0·01 +14·89 2 50	+ 0·05 +15·00 0 0
0·5	− 7·50 + 7·50 45 0	− 6·75 + 6·75 45 0	− 6·00 + 6·00 45 0	− 5·25 + 5·25 45 0	− 4·50 + 4·50 45 0	− 3·75 + 3·75 45 0	− 3·00 + 3·00 45 0	− 2·25 + 2·25 45 0	− 1·50 + 1·50 45 0	− 0·75 + 0·75 45 0	0·00 0·00
0·4	− 7·22 + 7·18 44 50	− 8·08 + 5·18 51 10	− 9·08 + 3·63 57 30	−10·16 + 2·46 63 30	−11·26 + 1·62 69 0	−12·31 + 1·01 73 40	−13·23 + 0·58 77 40	−13·99 + 0·29 81 10	−14·54 + 0·10 84 20	−14·89 − 0·01 87 10	−15·00 − 0·05 90 0
0·3	− 6·34 + 6·26 44 50	− 9·22 + 3·44 58 10	−12·80 + 1·91 68 20	−16·49 + 1·10 75 0	−19·92 + 0·64 79 10	−22·94 + 0·35 82 10	−25·45 + 0·17 84 20	−27·43 + 0·05 86 0	−28·86 − 0·03 87 30	−29·72 − 0·07 88 50	−30·00 − 0·08 90 0
0·2	− 4·85 + 4·75 44 40	−10·37 + 1·72 67 10	−17·07 + 0·73 77 10	−23·43 + 0·39 81 50	−29·09 + 0·19 84 20	−33·92 + 0·08 86 0	−37·90 0·00 87 10	−41·00 − 0·05 88 0	−43·22 − 0·08 88 40	−44·56 − 0·09 89 20	−45·00 − 0·10 90 0
0·1	− 2·74 + 2·66 44 40	−11·90 + 0·43 78 20	−21·82 + 0·14 84 20	−30·72 + 0·05 86 30	−38·47 0·00 87 30	−45·04 − 0·03 88 20	−50·42 − 0·05 88 50	−54·61 − 0·06 89 10	−57·61 − 0·07 89 30	−59·40 − 0·07 89 40	−60·00 − 0·07 90 0
0·0	0·00 0·00	−14·25 0·00 90 0	−27·00 0·00 90 0	−38·25 0·00 90 0	−48·00 0·00 90 0	−56·25 0·00 90 0	−63·00 0·00 90 0	−68·25 0·00 90 0	−72·00 0·00 90 0	−74·25 0·00 90 0	−75·00 0·00 90 0

The numbers from $v=0·0$ to $v=1·0$ apply also from $v=2·0$ to $v=1·0$, changing the sign of the angle. In regard to the pressure on the piers, where $v=0·0$ and $w=0·0$, see article 24.

Table III.—Strains on the interior points of a beam whose two ends rest upon piers, and which supports at the middle of its length a weight equal to half the weight of the beam. The length of the beam is supposed to be ten times its depth. The two numbers in each division of the Table are the values of the two principal strains, the unit being the depth of the beam. The positive sign denotes compression, and the negative sign tension. The angle is that by which the first-written strain is inclined to y, in the direction of diminishing x for increase of y. The direction of the second-written strain is at right angles to that of the first-written strain.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from one pier, to half the length of the beam).										
	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	0·00 0·00 ° ′	0·00 +21·75 0° 0′	0·00 +42·00 0° 0′	0·00 +60·75 0° 0′	0·00 +78·00 0° 0′	0·00 +93·75 0° 0′	0·00 +108·00 0° 0′	0·00 +120·75 0° 0′	0·00 +132·00 0° 0′	0·00 +141·75 0° 0′	0·00 +150·00 0° 0′
0·9	− 4·01 + 4·09 45 10	− 0·72 +18·19 11 50	− 0·29 +33·96 5 50	− 0·14 +48·82 3 50	− 0·07 +62·54 2 40	− 0·03 +75·10 2 0	+ 0·01 + 86·47 1 40	+ 0·02 + 96·65 1 20	+ 0·04 +105·64 1 0	+ 0·05 +113·42 0 50	+ 0·06 +120·02 0 40
0·8	− 7·15 + 7·25 45 10	− 2·76 +15·91 23 0	− 1·37 +26·67 13 10	− 0·80 +37·34 8 50	− 0·49 +47·39 6 20	− 0·31 +56·66 4 50	− 0·19 + 65·09 3 50	− 0·11 + 72·65 3 0	− 0·05 + 79·34 2 30	0·00 + 85·15 2 0	+ 0·03 + 90·07 1 30
0·7	− 9·41 + 9·49 45 10	− 5·42 +14·21 32 0	− 3·26 +20·14 22 10	− 2·08 +26·47 16 0	− 1·42 +32·71 12 0	− 0·95 +38·53 9 20	− 0·65 + 43·93 7 20	− 0·44 + 48·82 5 50	− 0·28 + 53·17 4 40	− 0·17 + 56·95 3 50	− 0·08 + 60·17 3 0
0·6	−10·78 +10·82 45 0	− 8·11 +12·50 39 0	− 6·23 +14·47 33 0	− 4·45 +16·65 27 30	− 3·27 +18·92 22 50	− 2·40 +21·20 18 50	− 1·75 + 23·40 15 30	− 1·25 + 25·44 12 50	− 0·88 + 27·33 10 30	− 0·60 + 29·00 8 30	− 0·38 + 30·43 6 50
0·5	−11·25 +11·25 45 0	−10·50 +10·50 45 0	− 9·75 + 9·75 45 0	− 9·00 + 9·00 45 0	− 8·25 + 8·25 45 0	− 7·50 + 7·50 45 0	− 6·75 + 6·75 45 0	− 6·00 + 6·00 45 0	− 5·25 + 5·25 45 0	− 4·50 + 4·50 45 0	− 3·75 + 3·75 45 0
0·4	−10·82 +10·78 45 0	−12·51 + 8·11 51 0	−14·47 + 6·23 57 0	−16·65 + 4·45 62 30	−18·92 + 3·27 67 10	−21·20 + 2·40 71 10	− 23·40 + 1·75 74 30	− 25·44 + 1·25 77 10	− 27·33 + 0·88 79 30	− 29·00 + 0·60 81 30	− 30·42 + 0·38 83 10
0·3	− 9·49 + 9·41 44 50	−14·21 + 5·42 58 0	−20·14 + 3·26 67 50	−26·47 + 2·08 74 0	−32·71 + 1·42 78 0	−38·53 + 0·95 80 40	− 43·93 + 0·66 82 40	− 48·82 + 0·44 84 10	− 53·17 + 0·28 85 20	− 56·95 + 0·17 86 10	− 60·17 + 0·08 87 0
0·2	− 7·25 + 7·15 44 50	−15·91 + 2·76 67 0	−26·67 + 1·37 76 50	−37·34 + 0·80 81 10	−47·39 + 0·49 83 40	−56·66 + 0·31 85 10	− 65·09 + 0·19 86 10	− 72·65 + 0·11 87 0	− 79·34 + 0·05 87 30	− 85·15 0·00 88 0	− 90·07 − 0·04 88 30
0·1	− 4·09 + 4·01 44 50	−18·19 + 0·72 78 10	−33·96 + 0·29 84 10	−48·82 + 0·14 86 10	−62·54 + 0·07 87 20	−75·10 + 0·03 88 0	− 86·47 − 0·01 88 20	− 96·65 − 0·02 88 40	−105·64 − 0·04 89 0	−113·42 − 0·05 89 10	−120·02 − 0·06 89 20
0·0	0·00 0·00	−21·75 0·00 90 0	−42·00 0·00 90 0	−60·75 0·00 90 0	−78·00 0·00 90 0	−93·75 0·00 90 0	−108·00 0·00 90 0	−120·75 0·00 90 0	−132·00 0·00 90 0	−141·75 0·00 90 0	−150·00 0·00 90 0

The numbers from $v=0·0$ to $v=1·0$ apply also from $v=2·0$ to $v=1·0$, changing the sign of the angle.

TABLE IV. Part I.—Strains on the interior points of a beam whose two ends rest upon piers, and which supports at the middle of its first half-length a weight equal to half the weight of the beam. Continued in Table IV. Part II.

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam).										
	First part of the beam, to the supported weight.						Second part of the beam, beyond the supported weight (continued in next Table).				
	0·0	0·1	0·2	0·3	0·4	0·5	0·5	0·6	0·7	0·8	0·9
1·0	0·00 0·00 ° ′	0·00 +25·50 0° 0′	0·00 +49·50 0° 0′	0·00 +72·00 0° 0′	0·00 +93·00 0° 0′	0·00 +112·50 0° 0′	0·00 +112·50 0° 0′	0·00 +115·50 0° 0′	0·00 +117·00 0° 0′	0·00 +117·00 0° 0′	0·00 +115·50 0° 0′
0·9	− 4·69 + 4·76 45 10	− 0·86 +21·33 11 50	− 0·37 +40·04 6 0	− 0·19 +57·87 3 50	− 0·10 +74·58 2 50	− 0·05 + 90·13 2 10	+ 0·07 + 90·00 0 30	+ 0·07 + 92·40 0 20	+ 0·07 + 93·60 0 10	+ 0·07 + 93·60 − 0 10	+ 0·07 + 92·40 − 0 20
0·8	− 8·35 + 8·45 45 10	− 3·28 +18·68 23 10	− 1·67 +31·46 13 20	− 1·00 +44·30 9 0	− 0·65 +56·54 6 30	− 0·43 + 68·03 5 0	+ 0·08 + 67·52 1 0	+ 0·09 + 69·31 0 40	+ 0·10 + 70·20 0 10	+ 0·10 + 70·20 − 0 10	+ 0·09 + 69·31 − 0 40
0·7	−10·98 +11·07 45 10	− 6·42 +16·70 32 0	− 3·94 +23·82 22 20	− 2·58 +31·46 16 10	− 1·77 +39·06 12 20	− 1·26 + 46·34 9 40	+ 0·03 + 45·06 2 0	+ 0·06 + 46·22 1 10	+ 0·08 + 46·80 0 20	+ 0·08 + 46·80 − 0 20	+ 0·06 + 46·22 − 1 10
0·6	−12·58 +12·62 45 0	− 9·57 +14·72 39 0	− 7·23 +17·17 33 10	− 5·44 +19·89 27 40	− 4·11 +22·76 23 10	− 3·11 + 25·66 19 20	− 0·10 + 22·64 4 30	0·00 + 23·15 2 40	+ 0·04 + 23·41 0 50	+ 0·04 + 23·41 − 0 50	0·00 + 23·15 − 2 40
0·5	−13·12 +13·12 45 0	−12·38 +12·38 45 0	−11·63 +11·63 45 0	−10·88 +10·88 45 0	−10·12 +10·12 45 0	− 9·38 + 9·38 45 0	− 1·88 + 1·88 45 0	− 1·13 + 1·13 45 0	− 0·38 + 0·38 45 0	− 0·38 + 0·38 −45 0	− 1·13 + 1·13 −45 0
0·4	−12·62 +12·58 45 0	−14·72 + 9·57 51 0	−17·17 + 7·23 56 50	−19·89 + 5·44 62 20	−22·76 + 4·11 66 50	− 25·66 + 3·11 70 40	− 22·64 + 0·10 85 30	− 23·15 0·00 87 20	− 23·41 − 0·04 89 10	− 23·41 − 0·04 −89 10	− 23·15 0·00 −87 20
0·3	−11·07 +10·98 44 50	−16·70 + 6·42 58 0	−23·82 + 3·94 67 40	−31·46 + 2·58 73 50	−39·06 + 1·77 77 40	− 46·34 + 1·26 80 20	− 45·06 − 0·03 88 0	− 46·22 − 0·06 88 50	− 46·80 − 0·08 89 40	− 46·80 − 0·08 −89 40	− 46·22 − 0·06 −88 50
0·2	− 8·45 +8·35 44 50	−18·67 + 3·28 66 50	−31·46 + 1·67 76 40	−44·30 + 1·00 81 0	−56·54 + 0·65 83 30	− 68·03 + 0·43 85 0	− 67·52 − 0·08 89 0	− 69·31 − 0·09 89 20	− 70·20 − 0·10 89 50	− 70·20 − 0·10 −89 50	− 69·31 − 0·09 −89 20
0·1	− 4·76 + 4·69 44 50	−21·33 + 0·86 78 10	−40·04 +0·37 84 0	−57·87 + 0·19 86 10	−74·58 + 0·11 87 10	− 90·13 + 0·05 87 50	− 90·01 − 0·07 89 30	− 92·40 − 0·07 89 40	− 93·60 − 0·07 89 50	− 93·60 − 0·07 −89 50	− 92·40 − 0·07 −89 40
0·0	0·00 0·00	−25·50 0·00 90 0	−49·50 0·00 90 0	−72·00 0·00 90 0	−93·00 0·00 90 0	−112·50 0·00 90 0	−112·50 0·00 90 0	−115·50 0·00 90 0	−117·00 0·00 90 0	−117·00 0·00 −90 0	−115·50 0·00 −90 0

TABLE IV. Part II. (Continuation of Table IV. Part I.).—Strains on the interior points of a beam whose two ends rest upon piers, and which supports at the middle of its first half-length a weight equal to half the weight of the beam.

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam). Second part of the beam, beyond the supported weight (continued from last Table).										
	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
1·0	0·00 +112·50 0° 0′	0·00 +108·00 0° 0′	0·00 +102·00 0° 0′	0·00 +94·50 0° 0′	0·00 +85·50 0° 0′	0·00 +75·00 0° 0′	0·00 +63·00 0° 0′	0·00 +49·50 0° 0′	0·00 +34·50 0° 0′	0·00 +18·00 0° 0′	0·00 0·00 0° 0′
0·9	+ 0·07 + 90·00 − 0 30	+ 0·06 + 86·41 − 0 40	+ 0·05 + 81·62 − 0 50	+ 0·04 +75·63 − 1 10	+ 0·03 +68·45 − 1 30	0·00 +60·07 − 2 0	− 0·03 +50·51 − 2 40	− 0·09 +39·77 − 3 40	− 0·22 +29·79 − 5 50	− 0·57 +15·04 −11 40	−3·34 +3·41 −44 40
0·8	+ 0·08 + 67·52 − 1 0	+ 0·05 + 64·84 − 1 30	+ 0·02 + 61·28 − 2 0	− 0·03 +56·82 − 2 40	− 0·09 +51·49 − 3 30	− 0·19 +45·29 − 4 30	− 0·34 +38·24 − 6 10	− 0·59 +30·39 − 8 30	− 1·07 +21·87 −13 0	− 2·24 +13·14 −23 0	−5·95 +6·05 −44 50
0·7	+ 0·03 + 45·06 − 2 0	− 0·03 + 43·31 − 3 0	− 0·11 + 41·00 − 4 0	− 0·23 +38·12 − 5 10	− 0·40 +34·69 − 6 50	− 0·65 +30·73 − 8 50	− 1·01 +26·30 −11 30	− 1·59 +21·48 −15 40	− 2·59 +16·47 −22 0	− 4·43 +11·71 −32 0	−7·83 +7·91 −44 50
0·6	− 0·10 + 22·64 − 4 30	− 0·24 + 21·89 − 6 40	− 0·46 + 20·90 − 8 50	− 0·75 +19·70 −11 20	− 1·15 +18·30 −14 20	− 1·70 +16·75 −18 0	− 2·44 +15·09 −22 10	− 3·46 +13·40 −27 10	− 4·83 +11·77 −32 50	− 6·64 +10·29 −39 0	−8·98 +9·02 −45 0
0·5	− 1·88 + 1·88 −45 0	− 2·63 + 2·63 −45 0	− 3·38 + 3·38 −45 0	− 4·13 + 4·13 −45 0	− 4·88 + 4·88 −45 0	− 5·63 + 5·63 −45 0	− 6·32 + 6·32 −45 0	− 7·12 + 7·12 −45 0	− 7·88 + 7·88 −45 0	− 8·62 + 8·62 −45 0	−9·38 +9·38 −45 0
0·4	− 22·64 + 0·10 −85 30	− 21·89 + 0·24 −83 20	− 20·90 + 0·46 −81 10	−19·70 + 0·75 −78 40	−18·30 + 1·15 −75 40	−16·75 + 1·70 −72 0	−15·09 + 2·44 −67 50	−13·40 + 3·46 −62 50	−11·77 + 4·83 −57 10	−10·29 + 6·64 −51 0	−9·02 +8·98 −45 0
0·3	− 45·06 − 0·03 −88 0	− 43·31 + 0·03 −87 0	− 41·00 + 0·11 −86 0	−38·12 + 0·23 −84 50	−34·69 + 0·40 −83 10	−30·73 + 0·65 −81 10	−26·30 + 1·01 −78 30	−21·48 + 1·59 −74 20	−16·47 + 2·59 −68 0	−11·71 + 4·43 −58 0	−7·92 +7·83 −45 10
0·2	− 67·52 − 0·08 −89 0	− 64·84 − 0·05 −88 30	− 61·28 − 0·02 −88 0	−56·82 + 0·03 −87 20	−51·49 + 0·09 −86 30	−45·29 + 0·19 −85 30	−38·24 + 0·34 −83 50	−30·39 + 0·59 −81 30	−21·87 + 1·07 −77 0	−13·14 + 2·24 −67 0	−6·05 +5·95 −45 10
0·1	− 90·01 − 0·07 −89 30	− 86·41 − 0·06 −89 20	− 81·62 − 0·05 −89 10	−75·63 − 0·04 −88 50	−68·45 − 0·03 −88 30	−60·07 0·00 −88 0	−50·51 + 0·03 −87 20	−39·77 + 0·09 −86 20	−27·79 + 0·22 −84 10	−15·04 + 0·57 −78 20	−3·41 +3·34 −45 20
0·0	−112·50 0·00 −90 0	−108·00 0·00 −90 0	−102·00 0·00 −90 0	−94·50 0·00 −90 0	−85·50 0·00 −90 0	−75·00 0·00 −90 0	−63·00 0·00 −90 0	−49·50 0·00 −90 0	−34·50 0·00 −90 0	−18·00 0·00 −90 0	0·00 0·00

TABLE V.—Strains on the interior points of a beam whose ends rest upon piers, and in which a strain (of the nature of a moment or couple) is impressed on each end, as in the interior tubes of the Britannia Bridge.

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam). 0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	−37·50 0·00 90° 0′	−23·25 0·00 90° 0′	−10·50 0·00 90° 0′	0·00 + 0·75 0° 0′	0·00 +10·50 0° 0′	0·00 +18·75 0° 0′	0·00 +25·50 0° 0′	0·00 +30·75 0° 0′	0·00 +34·50 0° 0′	0·00 +36·75 0° 0′	0·00 +37·50 0° 0′
0·9	−30·24 + 0·31 84 50	−18·91 + 0·38 82 40	− 8·92 + 0·59 76 30	− 1·57 + 2·25 41 0	− 0·23 + 8·70 10 40	− 0·05 +15·12 5 10	+ 0·02 +20·46 3 0	+ 0·05 +24·63 1 50	+ 0·06 +27·61 1 10	+ 0·07 +29·40 0 30	+ 0·07 +30·00 0 0
0·8	−23·48 + 1·07 78 30	−15·17 + 1·32 74 10	− 8·10 + 1·90 64 50	− 3·09 + 3·64 43 30	− 1·04 + 7·43 21 30	− 0·40 +11·75 12 40	− 0·14 +15·54 7 10	− 0·02 +18·56 4 30	+ 0·05 +20·75 2 40	+ 0·09 +22·06 1 20	+ 0·10 +22·50 0 0
0·7	−17·29 + 2·37 70 0	−11·97 + 2·75 64 50	− 7·53 + 3·42 56 30	− 4·22 + 4·60 44 20	− 2·16 + 6·45 30 40	− 1·07 + 8·66 20 10	− 0·51 +10·79 13 10	− 0·22 +12·60 8 30	−0·03 +13·92 5 10	+ 0·06 +14·73 2 30	+ 0·08 +15·00 0 0
0·6	−11·86 + 4·40 58 50	− 9·19 + 4·59 55 0	− 6·89 + 4·83 50 20	− 4·94 + 5·14 44 40	− 3·37 + 5·51 38 20	− 2·15 + 5·95 31 20	− 1·26 + 6·41 24 20	− 0·64 + 6·84 17 40	− 0·24 + 7·19 11 20	− 0·02 + 7·42 5 30	+ 0·05 + 7·50 0 0
0·5	− 7·50 + 7·50 45 0	− 6·75 + 6·75 45 0	− 6·00 + 6·00 45 0	− 5·25 + 5·25 45 0	− 4·50 + 4·50 45 0	− 3·75 + 3·75 45 0	− 3·00 + 3·00 45 0	− 2·25 + 2·25 45 0	− 1·50 + 1·50 45 0	− 0·75 + 0·75 45 0	0·00 0·00
0·4	− 4·40 +11·86 31 10	− 4·59 + 9·19 35 0	− 4·83 + 6·89 39 40	− 5·14 + 4·94 45 20	− 5·51 + 3·37 51 40	− 5·95 + 2·15 58 40	− 6·41 + 1·26 65 40	− 6·84 + 0·64 72 20	− 7·19 + 0·24 78 40	− 7·42 + 0·02 84 30	− 7·50 − 0·05 90 0
0·3	− 2·34 +17·29 20 0	− 2·75 +11·97 25 10	− 3·42 + 7·53 33 30	− 4·60 + 4·22 45 40	− 6·45 + 2·16 59 20	− 8·66 + 1·07 69 50	−10·79 + 0·51 76 50	−12·60 + 0·22 81 30	−13·92 + 0·03 84 50	−14·73 − 0·06 87 30	−15·00 − 0·08 90 0
0·2	− 1·07 +23·48 11 30	− 1·32 +15·17 15 50	− 1·90 + 8·10 25 10	− 3·64 + 3·09 46 30	− 7·43 + 1·04 68 30	−11·75 + 0·40 77 20	−15·54 + 0·14 82 50	−18·56 + 0·02 85 30	−20·75 − 0·05 87 20	−22·06 − 0·09 88 40	−22·50 − 0·10 90 0
0·1	− 0·31 +30·24 5 10	− 0·38 +18·91 7 20	− 0·59 + 8·92 13 30	− 2·25 + 1·57 49 0	− 8·70 + 0·23 79 20	−15·12 + 0·05 84 50	−20·46 − 0·02 87 0	−24·63 − 0·05 88 10	−27·61 − 0·06 88 50	−29·40 − 0·07 89 30	−30·00 − 0·07 90 0
0·0	0·00 +37·50 0 0	0·00 +23·25 0 0	0·00 +10·50 0 0	− 0·75 0·00 90 0	−10·50 0·00 90 0	−18·75 0·00 90 0	−25·50 0·00 90 0	−30·75 0·00 90 0	−34·50 0·00 90 0	−36·75 0·00 90 0	−37·50 0·00 90 0

The numbers from $v=0·0$ to $v=1·0$ apply also from $v=2·0$ to $v=1·0$, changing the sign of the angle.

Table VI. Part I.—Strains on the interior points of a beam whose ends rest upon piers, and on one end of which a strain (of the nature of a moment or couple) is impressed, as in the exterior tubes of the Britannia Bridge. (Continued in Part II.)

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam). The strain is impressed where $v=0$. 0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	−37·50 0·00 90° 0′	−21·38 0·00 90° 0′	− 6·75 0·00 90° 0′	0·00 + 6·38 0° 0′	0·00 +18·00 0° 0′	0·00 +28·13 0° 0′	0·00 +36·75 0° 0′	0·00 +43·88 0° 0′	0·00 +49·50 0° 0′	0·00 +53·63 0° 0′	0·00 +56·25 0° 0′
0·9	−30·30 + 0·38 84 20	−17·54 + 0·51 81 0	− 6·37 + 1·04 68 50	− 0·77 + 5·95 20 50	− 0·19 +14·66 7 40	− 0·05 +22·63 4 20	0·00 +29·47 2 50	+ 0·03 +35·14 1 50	+ 0·05 +39·62 1 20	+ 0·06 +42·91 0 50	+ 0·07 +45·00 0 30
0·8	−23·72 + 1·32 77 10	−14·48 + 1·76 71 20	− 6·88 + 2·92 57 30	− 2·15 + 6·34 32 20	− 0·94 +11·83 16 20	− 0·43 +17·40 9 50	− 0·19 +22·34 6 30	− 0·06 +26·48 4 30	+ 0·01 +29·78 3 0	+ 0·06 +32·21 2 0	+ 0·09 +33·76 1 0
0·7	−17·81 + 2·89 68 20	−12·00 + 3·54 61 50	− 7·30 + 4·68 51 40	− 4·02 + 6·66 38 20	− 2·15 + 9·43 26 0	− 1·17 +12·50 17 40	− 0·63 +15·41 12 10	− 0·32 +17·95 8 30	− 0·13 +20·01 5 50	− 0·01 +21·54 3 50	+ 0·06 +22·53 2 0
0·6	−12·66 + 5·21 57 30	− 9·80 + 5·58 53 10	− 7·35 + 6·05 48 0	− 5·31 + 6·63 42 0	− 3·69 + 7·34 35 40	− 2·46 + 8·13 29 10	− 1·56 + 8·95 23 0	− 0·92 + 9·74 17 30	− 0·48 +10·43 12 40	− 0·19 +10·97 8 30	− 0·02 +11·32 4 30
0·5	− 8·44 + 8·44 45 0	− 7·69 + 7·69 45 0	− 6·94 + 6·94 45 0	− 6·19 + 6·19 45 0	− 5·44 + 5·44 45 0	− 4·69 + 4·69 45 0	− 3·94 + 3·94 45 0	− 3·19 + 3·19 45 0	− 2·44 + 2·44 45 0	− 1·69 + 1·69 45 0	− 0·94 + 0·94 45 0
0·4	− 5·21 +12·66 32 30	− 5·58 + 9·80 36 50	− 6·05 + 7·35 42 0	− 6·63 + 5·31 48 0	− 7·34 + 3·69 54 20	− 8·13 + 2·46 60 50	− 8·95 + 1·56 67 0	− 9·74 + 0·92 72 30	−10·43 + 0·48 77 20	−10·97 + 0·19 81 30	−11·32 + 0·02 85 30
0·3	− 2·89 +17·81 21 40	− 3·54 +12·00 28 10	− 4·68 + 7·30 38 20	− 6·66 + 4·03 51 40	− 9·43 + 2·15 64 0	−12·50 + 1·17 72 20	−15·41 + 0·61 77 50	−17·95 + 0·32 81 30	−20·01 + 0·13 84 10	−21·54 + 0·01 86 10	−22·53 − 0·06 88 0
0·2	− 1·32 +23·72 12 50	− 1·76 +14·48 18 40	− 2·92 + 6·88 32 30	− 6·34 + 2·15 57 40	−11·83 + 0·94 73 30	−17·40 + 0·43 80 10	−22·34 + 0·19 83 30	−26·48 + 0·06 85 30	−29·78 − 0·01 87 0	−32·21 − 0·06 88 0	−33·76 − 0·09 89 0
0·1	− 0·38 +30·30 5 40	− 0·51 +17·54 9 0	− 1·04 + 6·37 21 10	− 5·95 + 0·77 69 10	−14·66 + 0·19 82 20	−22·63 + 0·05 85 40	−29·47 0·00 87 10	−35·14 − 0·03 88 10	−39·62 − 0·05 88 40	−42·91 − 0·06 89 10	−45·00 − 0·07 89 30
0·0	0·00 +37·50 0 0	0·00 +21·38 0 0	0·00 + 6·75 0 0	− 6·38 0·00 90 0	−18·00 0·00 90 0	−28·13 0·00 90 0	−36·75 0·00 90 0	−43·88 0·00 90 0	−49·50 0·00 90 0	−53·63 0·00 90 0	−56·25 0·00 90 0

TABLE VI. Part II.—Strains on the interior points of a beam whose ends rest upon piers, and on one end of which a strain (of the nature of a moment or couple) is impressed, as in the exterior tubes of the Britannia Bridge. (Continued from Part I.)

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam). No strain is impressed where $v=2{\cdot}0$.										
	1·1	$\frac{9}{8}$	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
1·0	0·00 +57·38 0° 0′	0·00 +57·42 0° 0′	0·00 +57·00 0° 0′	0·00 +55·13 0° 0′	0·00 +51·75 0° 0′	0·00 +46·88 0° 0′	0·00 +40·50 0° 0′	0·00 +32·63 0° 0′	0·00 +23·25 0° 0′	0·00 +12·38 0° 0′	0·00 0·00 ° ′
0·9	+ 0·07 +45·90 0 10	+ 0·07 +45·94 0 0	+ 0·07 +45·60 − 0 20	+ 0·07 +44·11 − 0 40	+ 0·06 +41·41 − 1 0	+ 0·04 +37·53 − 1 30	+ 0·02 +32·45 − 2 20	− 0·02 +26·19 − 3 20	− 0·11 +18·78 − 5 30	− 0·36 +10·33 −11 30	− 2·33 + 2·40 −45 30
0·8	+ 0·10 +34·43 0 10	+ 0·10 +34·45 0 0	+ 0·09 +34·20 − 0 40	+ 0·08 +33·10 − 1 30	+ 0·04 +31·11 − 2 30	− 0·02 +28·24 − 3 40	− 0·12 +24·51 − 5 20	− 0·29 +19·96 − 7 50	− 0·62 +14·67 −12 30	− 1·46 + 8·98 −22 40	− 4·15 + 4·25 −45 20
0·7	+ 0·08 +22·95 0 20	+ 0·08 +22·97 0 0	+ 0·07 +22·81 − 1 10	+ 0·03 +22·11 − 3 0	− 0·06 +20·84 − 4 50	− 0·21 +19·04 − 7 10	− 0·45 +16·74 −10 10	− 0·86 +13·99 −14 40	− 1·58 +10·96 −21 20	− 2·94 + 7·97 −31 50	− 5·47 + 5·56 −45 10
0·6	+ 0·05 +11·48 0 50	+ 0·05 +11·48 0 0	+ 0·02 +11·43 − 2 40	− 0·10 +11·17 − 6 30	− 0·32 +10·72 −10 30	− 0·68 +10·10 −15 0	− 1·21 + 9·36 −20 10	− 1·97 + 8·54 −26 0	− 3·03 + 7·73 −32 20	− 4·45 + 6·97 −38 50	− 6·28 + 6·32 −45 10
0·5	− 0·19 + 0·19 45 0	0·00 0·00	− 0·56 + 0·56 −45 0	− 1·31 + 1·31 −45 0	− 2·06 + 2·06 −45 0	− 2·81 + 2·81 −45 0	− 3·56 + 3·56 −45 0	− 4·31 + 4·31 −45 0	− 5·06 + 5·06 −45 0	− 5·81 + 5·81 −45 0	− 6·56 + 6·56 −45 0
0·4	−11·48 − 0·05 89 10	−11·48 − 0·05 ±90 0	−11·43 − 0·02 −87 20	−11·17 + 0·10 −83 30	−10·72 + 0·32 −79 30	−10·10 + 0·68 −75 0	− 9·36 + 1·21 −69 50	− 8·54 + 1·97 −64 0	− 7·73 + 3·03 −57 40	− 6·97 + 4·45 −51 10	− 6·32 + 6·28 −44 50
0·3	−22·95 − 0·08 89 40	−22·97 − 0·08 90 0	−22·81 − 0·07 −88 50	−22·11 − 0·03 −87 0	−20·84 + 0·06 −85 10	−19·04 + 0·21 −82 50	−16·74 + 0·45 −79 50	−13·99 + 0·86 −75 20	−10·96 + 1·58 −68 40	− 7·97 + 2·94 −58 10	− 5·56 + 5·47 −44 50
0·2	−34·42 − 0·10 89 50	−34·45 − 0·10 90 0	−34·20 − 0·09 −89 20	−33·10 − 0·08 −88 30	−31·11 − 0·04 −87 30	−28·24 + 0·02 −86 20	−24·51 + 0·12 −84 40	−19·96 + 0·29 −82 10	−14·67 + 0·62 −77 30	− 8·98 + 1·46 −67 20	− 4·25 + 4·15 −44 40
0·1	−45·90 − 0·07 89 50	−45·94 − 0·07 90 0	−45·60 − 0·07 −89 40	−44·11 − 0·07 −89 20	−41·41 − 0·06 −89 0	−37·53 − 0·04 −88 30	−32·45 − 0·02 −87 40	−26·19 + 0·02 −86 40	−18·78 + 0·11 −84 30	−10·33 + 0·36 −78 30	− 2·40 + 2·33 −44 30
0·0	−57·38 0·00 90 0	−57·42 0·00 90 0	−57·00 0·00 −90 0	−55·13 0·00 −90 0	−51·75 0·00 −90 0	−46·88 0·00 −90 0	−40·50 0·00 −90 0	−32·63 0·00 −90 0	−23·25 0·00 −90 0	−12·38 0·00 −90 0	0·00 0·00

Theory of Strains, article 6.

Figure 1.

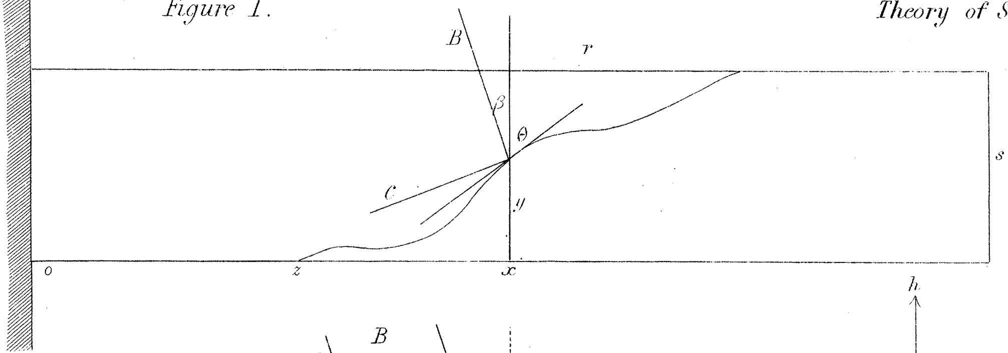

Figure 2.
article 7.

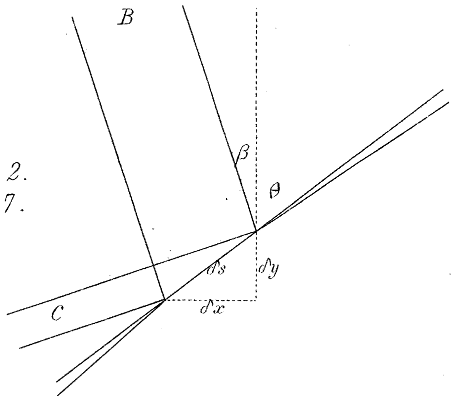

Figure 3, article 10.

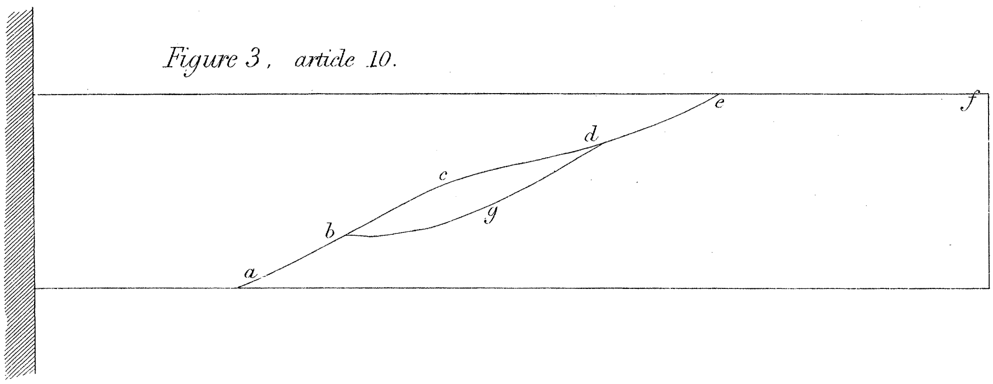

Figure 4.

Strains in the interior of a beam which projects from a wall.

(The continuous curves indicate the direction of thrust or compression; the interrupted curves or chain-lines indicate the direction of pull or tension.) For magnitudes and directions of strains, see Table 1.

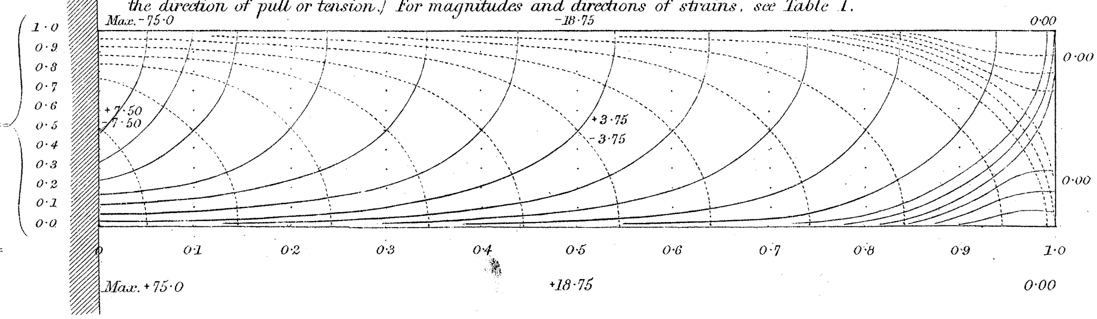

Figure 5. **Theory of Strains.**

Strains in the interior of a beam, whose ends rest upon piers, and which supports no other weight.

(The continuous curves indicate the direction of thrust or compression; and the interrupted curves or chain-lines indicate the direction of pull or tension.)

For magnitudes and directions of strains, see Table II.

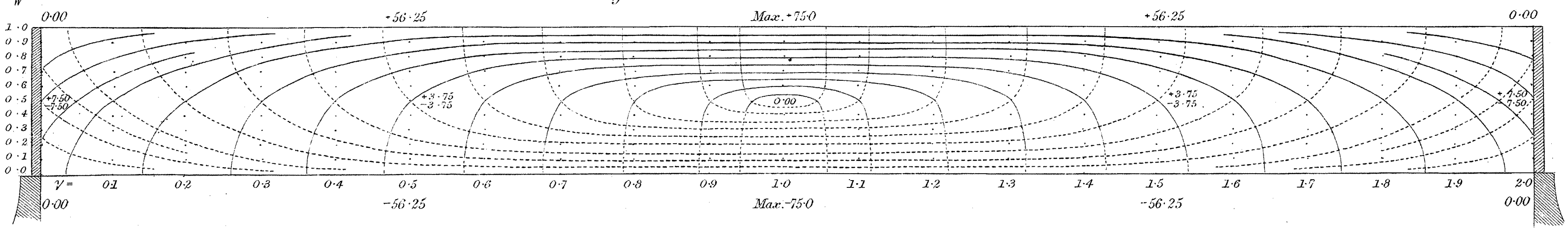

Figure 6.

Strains in the interior of a beam, whose ends rest upon piers, and which supports, at the middle of its length, a weight equal to half the weight of the beam.

For magnitudes and directions of strains, see Table III. *The curves which unite in the ordinate V = 1·0 meet at a small angle.*

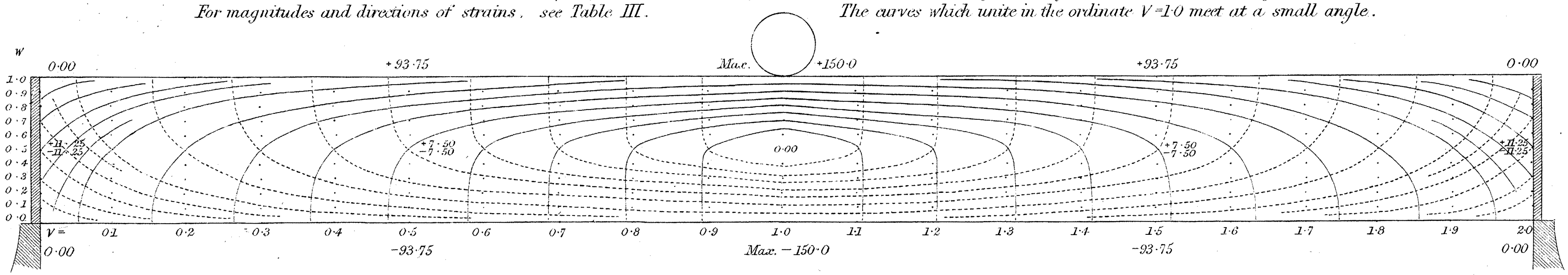

Figure 7.

Strains in the interior of a beam, whose ends rest upon piers, and which supports, at the middle of a half-length, a weight equal to half the weight of the beam.

For magnitudes and directions of strains, see Table IV. Parts I. and II.

The curves which unite in the ordinate V = 0·5 meet at a small angle.

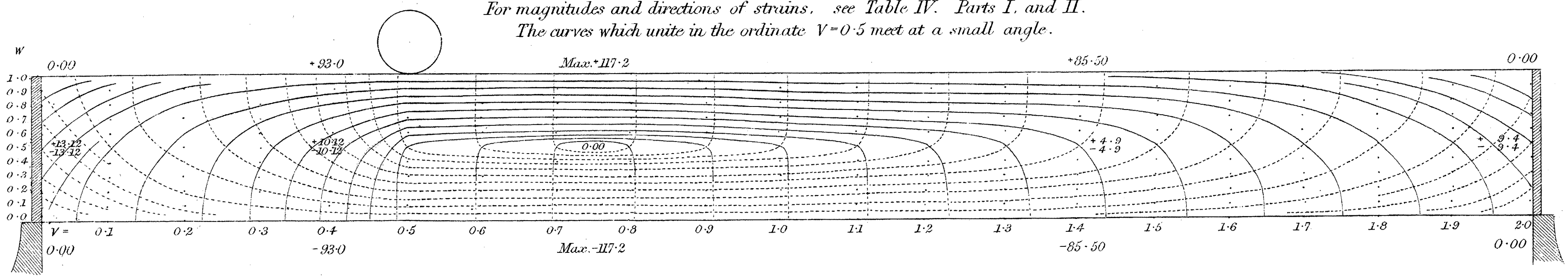

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Figure 8.

Theory of Strains.

Strains in the interior of a beam, whose ends rest upon piers, and in which a strain (of the nature of a moment or couple) is impressed on each end, as in the interior tubes of the Britannia Bridge. For magnitudes and directions of strains, see Table V.

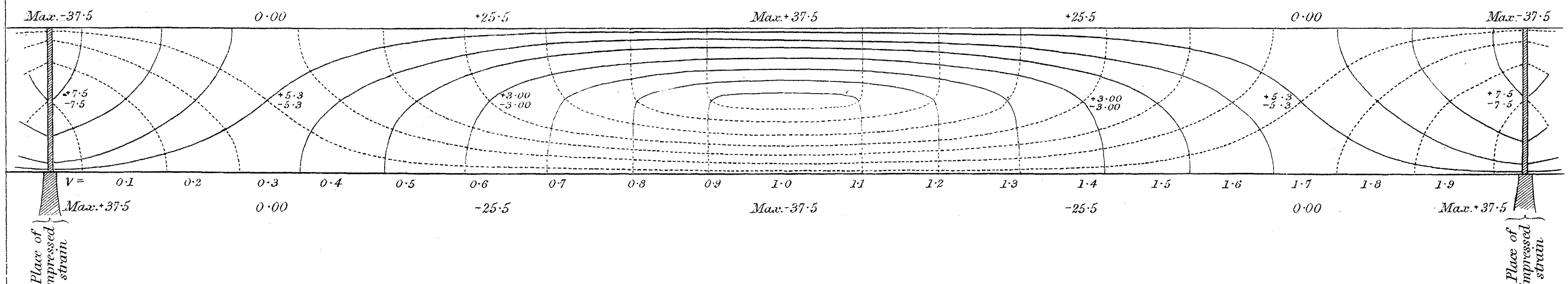
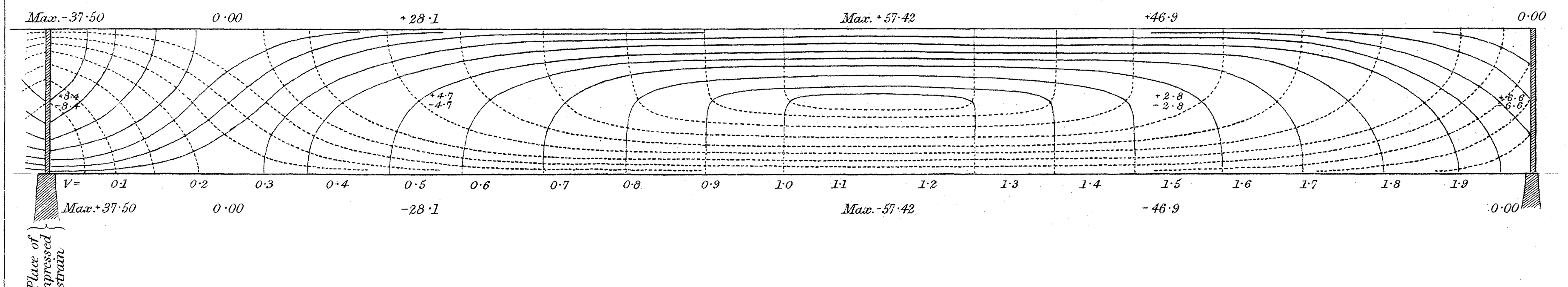


Figure 9.

Strains in the interior of a beam, whose ends rest upon piers, and in which a strain (of the nature of a moment or couple) is impressed on one end, as in the exterior tubes of the Britannia Bridge. For magnitudes and directions of strains, see Table VI. parts I and II.



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